Brief paper

# Stabilization of linear strict-feedback systems with delayed integrators☆

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## ABSTRACT

The problem of compensation of input delays for unstable linear systems was solved in the late 1970s. Systems with simultaneous input and state delay have remained a challenge, although exponential stabilization has been solved for systems that are not exponentially unstable, such as chains of delayed integrators and systems in the 'feedforward' form. We consider a general system in strict-feedback form with delayed integrators, which is an example of a particularly challenging class of exponentially unstable systems with simultaneous input and state delays, and design a predictor feedback controller for this class of systems. Exponential stability is proven with the aid of a Lyapunov–Krasovskii functional that we construct using the PDE backstepping approach.

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## 1. Introduction

Stabilization of linear systems with input delays continues to be an active area of research. Various control schemes for systems with input delay have been developed, with the starting point for many of them being the Smith predictor (Smith, 1959). The most important extensions of the Smith predictor have been designs based on the finite spectrum assignment framework (Artstein, 1982; Fiagbedzi & Pearson, 1986; Jankovic, 2009a, 2010; Kwon & Pearson, 1980; Manitius & Olbrot, 1979; Mondie & Michiels, 2003; Olbrot, 1978; Richard, 2003; Zhong, 2006). In addition to these designs, adaptive versions of predictor-based linear controllers are proposed in Evesque, Annaswamy, Niculescu, and Dowling (2003a), Liu and Krstic (2001), Niculescu and Annaswamy (2003b), Zhou, Wang, and Wen (2008), whereas adaptive controllers for unknown delay have been developed recently in Bekiaris-Liberis and Krstic (2010), Bresch-Pietri and Krstic (2009a,b) and Yildiray, Annaswamy, Kolmanovsky, and Yanakiev (2010). Moreover, various control designs for nonlinear systems exist (Jankovic, 2001, 2003, 2009b; Karafyllis, 2006; Krstic, 2008a, 2010; Mazenc & Bliman, 2004; Mazenc, Mondie, & Francisco, 2004).

Despite the fact that numerous papers deal with linear systems with input delay, the problem of controller design for systems with simultaneous input and state delay has been tackled in only a few Refs. Fiagbedzi and Pearson (1986), Jankovic (2009a), Jankovic (2010), Loiseau (2000), Manitius and Olbrot (1979) and Watanabe, Nobuyama, Kitamori, and Ito (1992). In this work we consider a specially chosen possibly open-loop unstable linear system, with the special form of a strict-feedback system with delayed integrators. Specifically, we consider the following *n*-dimensional linear system

$$\dot{\bar{X}}_1(t) = \bar{a}_{11}\bar{X}_1(t) + b_1\bar{X}_2(t - D_1) \tag{1}$$

$$\dot{\bar{X}}_2(t) = \bar{a}_{21}\bar{X}_1(t) + \bar{a}_{22}\bar{X}_2(t) + b_2\bar{X}_3(t - D_2) \tag{2}$$

$$\vdots$$

$$\dot{\bar{X}}_n(t) = \bar{a}_{n1}\bar{X}_1(t) + \cdots + \bar{a}_{nn}\bar{X}_n(t) + b_n\bar{U}(t - D_n), \tag{3}$$

where $\bar{X}_i(t), \bar{a}_{ij}, \bar{U}(t) \in \mathbb{R}$, $b_i \neq 0$, and $D_i \in \mathbb{R}_+$.

For this system we develop a predictor-based controller (Section 2). To achieve this we use tools from the boundary control of first order linear hyperbolic PDEs (Krstic & Smyshlyaev, 2008a), together with the classical backstepping procedure (Krstic, Kanellakopoulos, & Kokotovic, 1995). Specifically, an infinite dimensional backstepping transformation is used, together with a control law, to convert the system to an exponentially stable (in a certain sense) target system. Using the boundness of the backstepping transformation and its inverse, we then prove exponential stability of the closed-loop system using a suitably weighted Lyapunov–Krasovskii functional (Section 3). The effectiveness of the proposed controller is illustrated by a simulation example of a second order unstable system (Section 4).

## 2. Controller design

We start by redefining the states of system (1)–(3) such that the coefficients in front of the delayed terms are unity. That is, we

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define

$$X_1(t) = \bar{X}_1(t) \tag{4}$$

$$X_2(t) = b_1\bar{X}_2(t) \tag{5}$$

$$X_3(t) = b_1b_2\bar{X}_3(t) \tag{6}$$

$$\vdots$$

$$X_n(t) = b_1b_2\ldots b_{n-1}\bar{X}_n(t). \tag{7}$$

Moreover, for notational consistency we define

$$U(t) = b_1b_2\ldots b_n\bar{U}(t) \tag{8}$$

$$a_{ij} = \begin{cases} \bar{a}_{ij}, & \text{if } i = j \\ b_j\ldots b_{i-1}\bar{a}_{ij}, & \text{if } i > j \end{cases}. \tag{9}$$

In the new variables, system (1)–(3) is transformed to

$$\dot{X}_1(t) = a_{11}X_1(t) + X_2(t - D_1) \tag{10}$$

$$\dot{X}_2(t) = a_{21}X_1(t) + a_{22}X_2(t) + X_3(t - D_2) \tag{11}$$

$$\vdots$$

$$\dot{X}_n(t) = a_{n1}X_1(t) + \cdots + a_{nn}X_n(t) + U(t - D_n). \tag{12}$$

We state here our controller and in Section 3 we analyze the stability properties of the closed-loop system. The controller for the system (10)–(12) is given by

$$\begin{aligned}
U(t) &= u(D_n, t) \\
&= \alpha_n(D_n, t) \\
&= -a_{n1}P_1\left(t - \sum_{k=1}^{n-1} D_k\right) - \cdots - a_{nn}P_n(t) \\
&\quad - c_n\left(P_n(t) - \alpha_{n-1}\left(D_{n-1} + D_n, t\right)\right) \\
&\quad + \frac{\partial \alpha_{n-1}\left(D_{n-1} + D_n, t\right)}{\partial x},
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\alpha_i(x, t) &= -a_{i1}P_1\left(t - \sum_{k=1}^{n} D_k + x\right) - \cdots \\
&\quad - a_{ii}P_i\left(t - \sum_{k=i}^{n} D_k + x\right) \\
&\quad - c_i\left(P_i\left(t - \sum_{k=i}^{n} D_k + x\right) - \alpha_{i-1}(D_{i-1} + x, t)\right) \\
&\quad + \frac{\partial \alpha_{i-1}(D_{i-1} + x, t)}{\partial x}, \quad x \in \left[0, \sum_{k=i}^{n} D_k\right]
\end{aligned} \tag{14}$$

$$\begin{aligned}
\alpha_1(x, t) &= -(a_{11} + c_1)P_1\left(t + x - \sum_{k=1}^{n} D_k\right), \\
&\quad x \in \left[0, \sum_{k=1}^{n} D_k\right],
\end{aligned} \tag{15}$$

and the $c_i$, $i = 1, 2 \ldots n$ are arbitrary positive constants. In the above control scheme we use the $P_i(t)$ signals, the $\sum_{k=1}^{n} D_k$ seconds ahead predictors of the $X_i(t)$ state (this fact becomes clear later on). That is, it holds that $P_i(t) = X_i(t + \sum_{k=i}^{n} D_k)$. These signals are given by

$$P_1(t) = X_1(t) + \int_{t-\sum_{k=1}^{n} D_k}^{t} (a_{11}P_1(\theta) + P_2(\theta))d\theta \tag{16}$$

$$\begin{aligned}
P_2(t) &= X_2(t) + \int_{t-\sum_{k=2}^{n} D_k}^{t} (a_{21}P_1(\theta - D_1) \\
&\quad + a_{22}P_2(\theta) + P_3(\theta))d\theta
\end{aligned} \tag{17}$$

$$\vdots$$

$$\begin{aligned}
P_n(t) &= X_n(t) + \int_{t-D_n}^{t}\left(a_{n1}P_1\left(\theta - \sum_{k=1}^{n-1} D_k\right)\right. \\
&\quad \left. + a_{n2}P_2\left(\theta - \sum_{k=2}^{n-1} D_k\right) \cdots + a_{nn}P_n(\theta) + U(\theta)\right)d\theta,
\end{aligned} \tag{18}$$

with initial conditions

$$P_1(\theta) = X_1(0) + \int_{-\sum_{k=1}^{n} D_k}^{\theta} (a_{11}P_1(\sigma) + P_2(\sigma))d\sigma \tag{19}$$

$$\begin{aligned}
P_2(\theta) &= X_2(0) + \int_{-\sum_{k=2}^{n} D_k}^{\theta} (a_{21}P_1(\sigma - D_1) \\
&\quad + a_{22}P_2(\sigma) + P_3(\sigma))d\sigma
\end{aligned} \tag{20}$$

$$\vdots$$

$$\begin{aligned}
P_n(\theta) &= X_n(0) + \int_{-D_n}^{\theta}\left(a_{n1}P_1\left(\sigma - \sum_{k=1}^{n-1} D_k\right)\right. \\
&\quad \left. + a_{n2}P_2\left(\sigma - \sum_{k=2}^{n-1} D_k\right) \cdots + a_{nn}P_n(\sigma) + U(\sigma)\right)d\sigma,
\end{aligned} \tag{21}$$

where $\theta$ is defined in each $P_i(\theta)$ as $\theta \in [-\sum_{k=i}^{n} D_k, 0]$. Note here that the notation $\frac{\partial \alpha_{i-1}(D_{i-1}+x, t)}{\partial x}$ corresponds to $\frac{\partial \alpha_{i-1}(x', t)}{\partial x'}|_{x'=x+D_{i-1}}$ which includes the time derivatives of the signals $P_1(t), \ldots, P_{i-1}(t)$. These derivatives are obtained from (10)–(12) and (16)–(18).

## 3. Stability analysis

We first state a theorem describing our main stability result and then we prove it using a series of technical lemmas.

**Theorem 1.** *System* (10)–(12) *with the controller* (13) *is exponentially stable in the sense that there exist constants $\kappa$ and $\lambda$ such that*

$$\Omega(t) \leq \kappa\Omega(0)e^{-\lambda t}, \tag{22}$$

*where*

$$\begin{aligned}
\Omega(t) &= \frac{1}{2}\sum_{i=1}^{n} X_i^2(t) + \frac{1}{2}\sum_{i=2}^{n}\int_{t-D_{i-1}}^{t} X_i^2(\theta)d\theta \\
&\quad + \frac{1}{2}\int_{t-D_n}^{t} U^2(\theta)d\theta,
\end{aligned} \tag{23}$$

*and*

$$\int_{t-D_{i-1}}^{t} X_i^2(\theta)d\theta = \int_0^{D_{i-1}} \xi_i^2(x, t)dx = \|\xi_i(t)\|^2 \tag{24}$$

$$\int_{t-D_n}^{t} U^2(\theta)d\theta = \int_0^{D_n} u^2(x, t)dx = \|u(t)\|^2 \tag{25}$$

$$\xi_i(x, t) = X_i(t + x - D_{i-1}), \quad x \in [0, D_{i-1}] \tag{26}$$

$$u(x, t) = U(t + x - D_n), \quad x \in [0, D_n]. \tag{27}$$

We first give and prove the following lemmas.

**Lemma 1.** *The signals $P_i(t)$ defined in (16)–(18) are, respectively the $\sum_{k=i}^{n} D_k$ seconds ahead predictors of the $X_i(t)$ states. Moreover an equivalent representation for (16)–(18) is given by*

$$p_1\left(\sum_{k=1}^{n} D_k, t\right) = X_1(t) + \int_0^{\sum_{k=1}^{n} D_k} (a_{11}p_1(y, t) + p_2(y - D_1, t))dy \tag{28}$$

$$p_2\left(\sum_{k=2}^{n} D_k, t\right) = X_2(t) + \int_0^{\sum_{k=2}^{n} D_k} (a_{21}p_1(y, t) + a_{22}p_2(y, t) + p_3(y - D_2, t))dy \tag{29}$$

$$\vdots$$

$$p_n(D_n, t) = X_n(t) + \int_0^{D_n} (a_{n1}p_1(y, t) + \cdots + a_{nn}p_n(y, t) + u(y, t))dy, \tag{30}$$

*where*

$$p_i(x, t) = P_i\left(t + x - \sum_{k=i}^{n} D_k\right), \quad x \in \left[0, \sum_{k=i}^{n} D_k\right]. \tag{31}$$

**Proof.** Consider the equivalent representation of system (10)–(12) using transport PDEs for the delayed states and control

$$\dot{X}_1(t) = a_{11}X_1(t) + \xi_2(0, t) \tag{32}$$
$$\xi_{2_t}(x, t) = \xi_{2_x}(x, t) \tag{33}$$
$$\xi_2(D_1, t) = X_2(t) \tag{34}$$
$$\dot{X}_2(t) = a_{21}X_1(t) + a_{22}X_2(t) + \xi_3(0, t) \tag{35}$$
$$\xi_{3_t}(x, t) = \xi_{3_x}(x, t) \tag{36}$$
$$\xi_3(D_2, t) = X_3(t) \tag{37}$$
$$\vdots$$
$$\dot{X}_n(t) = a_{n1}X_1(t) + \cdots + a_{nn}X_n(t) + u(0, t) \tag{38}$$
$$u_t(x, t) = u_x(x, t) \tag{39}$$
$$u(D_n, t) = U(t). \tag{40}$$

Consider the following ODEs in $x$ (to become clear that these are ODEs in $x$, consider the time $t$ to act as a parameter rather as a running variable),

$$p_{1_x}(x, t) = a_{11}p_1(x, t) + p_2(x - D_1, t) \tag{41}$$
$$p_{2_x}(x, t) = a_{21}p_1(x, t) + a_{22}p_2(x, t) + p_3(x - D_2, t) \tag{42}$$
$$\vdots$$
$$p_{n_x}(x, t) = a_{n1}p_1(x, t) + \cdots + a_{nn}p_n(x, t) + u(x, t), \tag{43}$$

where, for each $p_i(x, t)$, $x$ varies in $[0, \sum_{k=i}^{n} D_k]$. The initial conditions for the above system of ODEs are given by

$$p_i(0, t) = X_i(t), \quad \forall i, \tag{44}$$

and

$$p_i(\theta_i, t) = X_i(t + \theta_i), \quad \theta_i \in [-D_{i-1}, 0], \ i = 2, \ldots, n. \tag{45}$$

Since system (41)–(43) is driven by the input $u(x, t)$, which satisfies a transport PDE, the same holds for all the $p_i(x, t)$ (see for example Krstic (2008a, 2010)). Thus,

$$\frac{\partial p_i(x, t)}{\partial t} = \frac{\partial p_i(x, t)}{\partial x}, \quad x \in \left[0, \sum_{k=i}^{n} D_k\right], \ \forall i. \tag{46}$$

By taking into account (44) we have that

$$p_i(x, t) = X_i(t + x), \quad x \in \left[0, \sum_{k=i}^{n} D_k\right], \ \forall i. \tag{47}$$

To see that (46) holds it is sufficient to prove that (47) is the unique solution of the ODEs in $x$ given by (41)–(43) with the initial conditions (44)–(45). Since then, the $p_i(x, t)$ are functions of only one variable, namely $x + t$, and consequently (46) holds. Thus, it remains to prove that (47) is the unique solution of the initial value problem (41)–(45). Toward this end, by taking into account (27) we point out that (47) satisfy the initial value problem (41)–(45). Then, assuming that $X_i(t + \theta_i)$, $i = 2, \ldots, n$ are continuous for all $\theta_i \in [-D_{i-1}, 0]$, using Theorem 2.1 from Hale and Verduyn Lunel (1993) we can conclude that (47) is the unique solution of the ODEs in $x$ given by (41)–(43) with the initial conditions (44)–(45). Thus, (46) holds.

From relation (47) becomes clear that the $p_i(x, t)$ are the $x$ seconds ahead predictors of the states. By defining

$$p_i\left(\sum_{k=i}^{n} D_k, t\right) = P_i(t), \quad \forall i, \tag{48}$$

we get (31). By integrating from 0 to $x$ (41)–(43) we get

$$p_1(x, t) = X_1(t) + \int_0^x (a_{11}p_1(y, t) + p_2(y - D_1, t))dy \tag{49}$$

$$p_2(x, t) = X_2(t) + \int_0^x (a_{21}p_1(y, t) + a_{22}p_2(y, t) + p_3(y - D_2, t))dy \tag{50}$$

$$\vdots$$

$$p_n(x, t) = X_n(t) + \int_0^x (a_{n1}p_1(y, t) + \cdots + a_{nn}p_n(y, t) + u(y, t))dy. \tag{51}$$

By setting in each $p_i(x, t)$, $x = \sum_{k=i}^{n} D_k$ and using (31) we get (28)–(30). □

It is important here to observe that the total delay from the input to each state $X_i(t)$ is $\sum_{k=i}^{n} D_k$. This explains the fact that our predictor intervals are different for each state and specifically must be $\sum_{k=i}^{n} D_k$ seconds for each state $X_i(t)$. Our controller design is based on a recursive procedure that transforms system (10)–(12) to a target system which is exponentially stable with the controller (13). Then, using the invertibility of this transformation, we prove exponential stability of the original system. We now state this transformation, along with its inverse.

**Lemma 2.** *The state transformation defined by*

$$Z_1(t) = X_1(t) \tag{52}$$
$$Z_{i+1}(t) = X_{i+1}(t) - \alpha_i(D_i, t), \quad i = 1, 2, \ldots, n-1, \tag{53}$$

*along with the transformation of the actuator state*

$$w(x, t) = u(x, t) - \alpha_n(x, t), \quad x \in [0, D_n], \tag{54}$$

*where the $\alpha_i(x, t)$ are defined as in (14)–(15), transforms the system (10)–(12) to the target system with the control law given by (13). The target system is given by*

$$\dot{Z}_1(t) = -c_1 Z_1(t) + Z_2(t - D_1) \tag{55}$$
$$\dot{Z}_2(t) = -c_2 Z_2(t) + Z_3(t - D_2) \tag{56}$$
$$\vdots$$
$$\dot{Z}_n(t) = -c_n Z_n(t) + W(t - D_n), \tag{57}$$

*where*

$$W(\theta) = 0, \quad \theta \geq 0. \tag{58}$$

**Proof.** Before we start our recursive procedure we rewrite the target system using transport PDEs as

$$\dot{Z}_1(t) = -c_1 Z_1(t) + \zeta_2(0, t) \tag{59}$$
$$\zeta_{2_t}(x, t) = \zeta_{2_x}(x, t) \tag{60}$$
$$\zeta_2(D_1, t) = Z_2(t) \tag{61}$$
$$\dot{Z}_2(t) = -c_2 Z_2(t) + \zeta_3(0, t) \tag{62}$$
$$\zeta_{3_t}(x, t) = \zeta_{3_x}(x, t) \tag{63}$$
$$\zeta_3(D_2, t) = Z_3(t) \tag{64}$$
$$\vdots$$
$$\dot{Z}_n(t) = -c_n Z_n(t) + w(0, t) \tag{65}$$
$$w_t(x, t) = w_x(x, t) \tag{66}$$
$$w(D_n, t) = 0. \tag{67}$$

Note that

$$\zeta_i(x, t) = Z_i(t + x - D_{i-1}), \quad x \in [0, D_{i-1}]. \tag{68}$$

*Step* 1. Following the backstepping procedure we first stabilize $X_1(t)$ with the virtual input $\alpha_1(D_1, t)$. We define

$$\zeta_2(x, t) = \xi_2(x, t) - \alpha_1(x, t), \tag{69}$$

then using (32) we get

$$\dot{X}_1(t) = a_{11} X_1(t) + \zeta_2(0, t) + \alpha_1(0, t). \tag{70}$$

By choosing $\alpha_1(x, t) = -(a_{11} + c_1) p_1(x, t)$ (note the equivalent representation of $\alpha_1(x, t)$ using (31)) and by using (44) we get

$$\dot{Z}_1(t) = -c_1 Z_1(t) + \zeta_2(0, t). \tag{71}$$

From (69) with $x = D_1$ and (61) it follows that

$$Z_2(t) = X_2(t) - \alpha_1(D_1, t). \tag{72}$$

By setting now

$$\zeta_3(x, t) = \xi_3(x, t) - \alpha_2(x, t), \tag{73}$$

and using (69), (33) and (35) we have

$$\dot{Z}_2(t) = \zeta_{2_x}(x, t)|_{x=D_1} = a_{21} X_1(t) + a_{22} X_2(t)$$
$$+ \zeta_3(0, t) + \alpha_2(0, t) - \frac{\partial \alpha_1(D_1, t)}{\partial x}, \tag{74}$$

where $\frac{\partial \alpha_1(D_1,t)}{\partial x}$ corresponds to $\frac{\partial \alpha_1(x,t)}{\partial x}|_{x=D_1}$ and we use the fact that $\alpha_{1_t}(x, t) = \alpha_{1_x}(x, t)$ (which is a consequence of relation (15)).

*Step* 2. By choosing

$$\alpha_2(x, t) = -a_{21} p_1(x, t) - a_{22} p_2(x, t) - c_2 (p_2(x, t)$$
$$- \alpha_1(D_1 + x, t)) + \frac{\partial \alpha_1(x + D_1, t)}{\partial x}$$
$$= -a_{21} p_1(x, t) - a_{22} p_2(x, t) - c_2 (p_2(x, t)$$
$$+ (a_{11} + c_1) p_1(D_1 + x, t))$$
$$- (a_{11} + c_1)(a_{11} p_1(x + D_1, t) + p_2(x, t)), \tag{75}$$

we get from (74) (with the help of (44)) that

$$\dot{Z}_2(t) = -c_2 Z_2(t) + \zeta_3(0, t). \tag{76}$$

By setting now $x = D_2$ in (73) and using (64) we get

$$Z_3(t) = X_3(t) - \alpha_2(D_2, t). \tag{77}$$

If we now define

$$\zeta_4(x, t) = \xi_4(x, t) - \alpha_3(x, t), \tag{78}$$

then with the help of (36) we get

$$\dot{Z}_3(t) = a_{31} X_1(t) + a_{32} X_2(t) + a_{33} X_3(t) + \zeta_4(0, t)$$
$$+ \alpha_3(0, t) - \frac{\partial \alpha_2(D_2, t)}{\partial x}. \tag{79}$$

*Step i.* Assume now that

$$\dot{Z}_{i-1}(t) = -c_{i-1} Z_{i-1}(t) + \zeta_i(0, t), \tag{80}$$

and define $\zeta_{i+1}(x, t)$ as

$$\zeta_{i+1}(x, t) = \xi_{i+1}(x, t) - \alpha_i(x, t). \tag{81}$$

Then from (53) with $x = D_{i-1}$ we have that

$$\dot{Z}_i(t) = a_{i1} X_1(t) + \cdots + a_{ii} X_i(t) + \zeta_{i+1}(0, t)$$
$$+ \alpha_i(0, t) - \frac{\partial \alpha_{i-1}(D_{i-1}, t)}{\partial x}. \tag{82}$$

Hence, with

$$\alpha_i(x, t) = -a_{i1} p_1(x, t) - \cdots - a_{ii} p_i(x, t)$$
$$- c_i(p_i(x, t) - \alpha_{i-1}(D_{i-1} + x, t)) + \frac{\partial \alpha_{i-1}(D_{i-1} + x, t)}{\partial x}, \tag{83}$$

we get

$$\dot{Z}_i(t) = -c_i Z_i(t) + \zeta_{i+1}(0, t). \tag{84}$$

*Step n.* In the last step we choose the controller $U(t)$. Since

$$\dot{Z}_n(t) = a_{n1} X_1(t) + \cdots + a_{nn} X_n(t) + u(0, t)$$
$$- \frac{\partial \alpha_{n-1}(D_{n-1}, t)}{\partial x}. \tag{85}$$

Then using (14) for $i = n$ we have that

$$\dot{Z}_n(t) = -c_n Z_n(t) + w(0, t), \tag{86}$$

and using (13)

$$w_t(x, t) = w_x(x, t) \tag{87}$$
$$w(D_n, t) = 0. \tag{88}$$

Assuming an initial condition for (87) as

$$w(x, 0) = w_0(x), \tag{89}$$

and by defining a new variable $W(\cdot)$ as

$$w_0(x) = W(x - D), \quad x \in [0, D_n], \tag{90}$$

we get that

$$w(x, t) = \begin{cases} W(t + x - D), & -D \leq t + x - D \leq 0 \\ 0, & t + x - D \geq 0 \end{cases}. \tag{91}$$

Defining $\theta = t + x - D$ one gets (58). Note here that based on (54), $w_0(x)$ is given by

$$w_0(x) = u(x, 0) - \alpha_n(x, 0), \quad x \in [0, D_n]. \quad \square \tag{92}$$

We now define the inverse transformation of (52)–(54).

**Lemma 3.** *The inverse transformation of* (52)–(54) *is defined as*

$$X_1(t) = Z_1(t) \tag{93}$$
$$X_{i+1}(t) = Z_{i+1}(t) + \beta_i(D_i, t), \quad i = 1, 2, \ldots, n - 1. \tag{94}$$
$$u(x, t) = w(x, t) + \beta_n(x, t), \quad x \in [0, D_n], \tag{95}$$

where the $\beta_i(x, t)$ are now given by

$$\beta_1(x, t) = -(a_{11} + c_1)\epsilon_1(x, t), \quad x \in \left[0, \sum_{k=1}^{n} D_k\right] \quad (96)$$

$$\begin{aligned}
\beta_i(x, t) &= -a_{i1}\epsilon_1(x, t) - a_{i2}(\epsilon_2(x, t) + \beta_1(D_1 + x, t)) \\
&\quad - \cdots - a_{ii}(\epsilon_i(x, t) + \beta_{i-1}(D_{i-1} + x, t)) \\
&\quad - c_i\epsilon_i(x, t) + \frac{\partial \beta_{i-1}(D_{i-1} + x, t)}{\partial x},
\end{aligned}$$
$$x \in \left[0, \sum_{k=i}^{n} D_k\right], \quad \forall i = 2, \ldots, n, \quad (97)$$

and the $\epsilon_i(x, t)$ (the predictors of the transformed states) are given by the following relations

$$\epsilon_1(x, t) = Z_1(t) + \int_0^x (-c_1\epsilon_1(y, t) + \epsilon_2(y - D_1, t))dy \quad (98)$$

$$\epsilon_2(x, t) = Z_2(t) + \int_0^x (-c_2\epsilon_2(y, t) + \epsilon_3(y - D_2, t))dy \quad (99)$$

$$\vdots$$

$$\epsilon_n(x, t) = Z_n(t) + \int_0^x (-c_n\epsilon_n(y, t) + w(y, t))dy, \quad (100)$$

where in each $\epsilon_i(x, t)$, $x$ varies in $[0, \sum_{k=i}^{n} D_k]$.

**Proof.** Applying similar arguments as in Lemma 2 we prove that the inverse transformation of (52)–(54) and (14)–(15) is given by (93)–(97). □

We now prove the stability of the transformed system.

**Lemma 4.** The target system is exponentially stable in the sense that there exist constants $M_1$, $m_1$ and $m_2$ such that

$$\Xi(t) \leq \frac{M_1(1 + D_{\max})}{m_2} \Xi(0)e^{-\frac{m_1}{M_1}t}, \quad (101)$$

where

$$\begin{aligned}
\Xi(t) &= \frac{1}{2}\sum_{i=1}^{n} Z_i^2(t) + \frac{1}{2}\sum_{i=2}^{n} \int_{t-D_{i-1}}^{t} Z_i^2(\theta)d\theta \\
&\quad + \frac{1}{2}\int_{t-D_n}^{t} W^2(\theta)d\theta
\end{aligned} \quad (102)$$

$$D_{\max} = \max\{D_i\}, \quad \forall i, \quad (103)$$

and

$$\int_{t-D_{i-1}}^{t} Z_i^2(\theta)d\theta = \int_0^{D_{i-1}} \zeta_i^2(x, t)dx = \|\zeta_i(t)\|^2 \quad (104)$$

$$\int_{t-D_n}^{t} W^2(\theta)d\theta = \int_0^{D_n} w^2(x, t)dx = \|w(t)\|^2. \quad (105)$$

**Proof.** We consider the following Lyapunov-like function

$$\begin{aligned}
V(t) &= \frac{1}{2}\sum_{i=1}^{n} k_i Z_i^2(t) + \frac{1}{2}\sum_{i=2}^{n} \lambda_i \int_0^{D_{i-1}} (1 + x)\zeta_i^2(x, t)dx \\
&\quad + \frac{\lambda_{n+1}}{2}\int_0^{D_n} (1 + x)w^2(x, t)dx.
\end{aligned} \quad (106)$$

Note that the above functional can be considered as a Control Lyapunov Functional in the sense of Karafyllis and Jiang (in press). This fact reinforces the strength of the present result: a Control Lyapunov Functional is actually constructed. By taking the time

derivative of the above function along the solutions of the $Z(t)$ system and by exploiting the fact that $\zeta_i(x, t)$ and $w(x, t)$ satisfy transport PDEs (based on (60), (63) and (66)), it follows that

$$\begin{aligned}
\dot{V}(t) &= -\sum_{i=1}^{n} c_i k_i Z_i^2(t) + \sum_{i=1}^{n-1} k_i Z_i(t)\zeta_{i+1}(0, t) \\
&\quad + k_n Z_n(t)w(0, t) + \frac{1}{2}\sum_{i=2}^{n} \lambda_i(1 + D_{i-1})Z_i^2(t) \\
&\quad - \frac{1}{2}\sum_{i=2}^{n} \lambda_i \zeta_i^2(0, t) - \frac{1}{2}\sum_{i=2}^{n} \lambda_i \int_0^{D_{i-1}} \zeta_i^2(x, t)dx \\
&\quad - \frac{\lambda_{n+1}}{2}w^2(0, t) - \frac{\lambda_{n+1}}{2}\int_0^{D_n} w^2(x, t)dx, \quad (107)
\end{aligned}$$

where we used integration by parts in the above integrals. By choosing the weights as

$$k_i = 2\frac{\lambda_i}{c_i}(1 + D_{i-1}), \quad i = 2, \ldots, n$$

$$k_1 = 2 \quad (108)$$

$$\lambda_i = 4\frac{\lambda_{i-1}(1 + D_{i-2})}{c_{i-1}^2}, \quad i = 3, \ldots, n+1$$

$$\lambda_2 = \frac{1}{2c_1}, \quad (109)$$

and after some manipulations that incorporate completion of squares we get

$$\begin{aligned}
\dot{V}(t) &\leq -\frac{1}{2}\sum_{i=1}^{n} c_i k_i Z_i^2(t) - \frac{1}{2}\sum_{i=2}^{n} \lambda_i \int_0^{D_{i-1}} \zeta_i^2(x, t)dx \\
&\quad - \frac{\lambda_{n+1}}{2}\int_0^{D_n} w^2(x, t)dx. \quad (110)
\end{aligned}$$

Defining

$$M_1 = \max\{k_i, \lambda_{i+1}\}, \quad i = 1, 2, \ldots, n. \quad (111)$$

$$m_1 = \min\left\{\frac{c_i k_i}{2}, \frac{\lambda_{i+1}}{2(1 + D_i)}\right\}, \quad i = 1, 2, \ldots, n, \quad (112)$$

it follows that

$$\dot{V}(t) \leq -\frac{m_1}{M_1}V(t). \quad (113)$$

If we now define

$$m_2 = \min\left\{\frac{k_i}{2}, \frac{\lambda_{i+1}}{2}\right\}, \quad i = 1, 2, \ldots, n, \quad (114)$$

then

$$\Xi(t) \leq \frac{V(0)}{m_2}e^{-\frac{m_1}{M_1}t} \leq \frac{M_1(1 + D_{\max})}{m_2}\Xi(0)e^{-\frac{m_1}{M_1}t}. \quad \square \quad (115)$$

We give now the following lemma which we prove in the Appendix.

**Lemma 5.** There exist constants $G_i$ such that

$$\begin{aligned}
p_i(x, t)^2 &\leq G_i\left(|X(t)|^2 + \sum_{i=2}^{n} \int_0^{D_{i-1}} \xi_i^2(y, t)dy \right. \\
&\quad \left. + \int_0^{D_n} u^2(y, t)dy\right), \quad \forall x \in \left[0, \sum_{k=i}^{n} D_k\right], \quad (116)
\end{aligned}$$

*where*

$$|X(t)|^2 = \sum_{i=1}^{n} X_i^2(t), \tag{117}$$

*and the bound* (116) *is independent of x.*

**Lemma 6.** *There exists a constant $\overline{M}$ such that*

$$\Xi(t) \le \overline{M}\Omega(t). \tag{118}$$

**Proof.** From (52)–(54) it follows that

$$Z_i^2(t) \le 2(X_i^2(t) + \alpha_{i-1}^2(D_i, t)), \quad i = 2, \dots, n \tag{119}$$

$$\zeta_i^2(x, t) \le 2(\xi_i^2(x, t) + \alpha_{i-1}^2(x, t)), \\ x \in [0, D_{i-1}], \; i = 2, \dots, n \tag{120}$$

$$w^2(x, t) \le 2(u^2(x, t) + \alpha_n^2(x, t)), \quad x \in [0, D_n]. \tag{121}$$

Moreover, from relations (14)–(15) one can see that the $\alpha_i(x, t)$ are linear functions of the predictors $p_1(x, t), \dots, p_i(x, t)$, hence it holds that

$$\alpha_i^2(x, t) \le b_i \sum_{k=1}^{i} p_k^2(x, t), \quad x \in \left[0, \sum_{k=i}^{n} D_k\right] \tag{122}$$

for some constants $b_i$. By employing the bound of Lemma 5, the lemma is proven. $\square$

**Lemma 7.** *There exist constants $F_i$ such that*

$$\epsilon_i^2(x, t) \le F_i \left( |Z(t)|^2 + \sum_{i=2}^{n} \int_0^{D_{i-1}} \zeta_i^2(y, t)dy \right.$$
$$\left. + \int_0^{D_n} w^2(y, t)dy \right), \quad x \in \left[0, \sum_{k=i}^{n} D_k\right]. \tag{123}$$

**Proof.** Immediately note that the relation for the $\epsilon_i(x, t)$ is similar to the relation for $p_i(x, t)$. Note here that in this case the derivation of the explicit bound is easier due to the special form of the $\epsilon_i(x, t)$ in (98)–(100). $\square$

**Lemma 8.** *There exists a constant $\underline{M}$ such that*

$$\underline{M}\Omega(t) \le \Xi(t). \tag{124}$$

**Proof.** Using relations (93)–(95) we get

$$X_i^2(t) \le 2(Z_i^2(t) + \beta_{i-1}^2(D_i, t)), \quad i = 2, \dots, n \tag{125}$$

$$\xi_i^2(x, t) \le 2(\zeta_i^2(x, t) + \beta_{i-1}^2(x, t)), \\ x \in [0, D_{i-1}], \; i = 2, \dots, n \tag{126}$$

$$u^2(x, t) \le 2(w^2(x, t) + \beta_n^2(x, t)), \quad x \in [0, D_n]. \tag{127}$$

By observing that $\beta_i(x, t)$ are linearly dependent on $\epsilon_1(x, t), \dots, \epsilon_i(x, t)$ we conclude that there exist constants $d_i$ such that

$$\beta_i^2(x, t) \le d_i \sum_{k=1}^{i} \epsilon_k^2(x, t), \quad x \in \left[0, \sum_{k=i}^{n} D_k\right]. \tag{128}$$

Using Lemma 7 the lemma is proven. $\square$

**Proof of Theorem 1.** Combining Lemmas 6 and 8 we have that

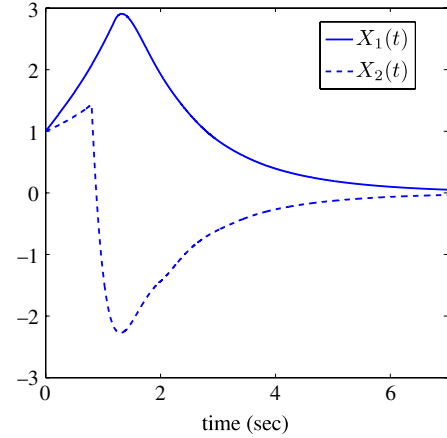$$\underline{M}\Omega(t) \le \Xi(t) \le \overline{M}\Omega(t). \tag{129}$$



**Fig. 1.** System's response for the simulation example.

Hence,

$$\Omega(t) \le \frac{\Xi(t)}{\underline{M}}, \tag{130}$$

and by Lemma 4 we get

$$\Omega(t) \le \frac{\overline{M}M_1(1 + D_{\max})}{\underline{M}m_2}\Omega(0)e^{-\frac{m_1}{M_1}t}. \tag{131}$$

Thus Theorem 1 is proven with

$$\kappa = \frac{\overline{M}M_1(1 + D_{\max})}{\underline{M}m_2} \tag{132}$$

$$\lambda = \frac{m_1}{M_1}. \quad \square \tag{133}$$

## 4. Simulations

We illustrate here our controller with a second order example with parameters $a_{11} = a_{21} = a_{22} = 0.2$, $D_1 = 0.4$, $D_2 = 0.8$ and $c_1 = c_2 = 2$. The initial conditions for the controller are given by (19)–(21) and for the system are $X_1(0) = X_2(0) = 1$ and $X_2(\theta) = 1, \theta \in [-D_1, 0]$. This system is unstable (to see this one can use Olgac and Sipahi (2002)). In the present case the controller will have the form

$$U(t) = u(D_2, t)$$
$$= \alpha_2(D_2, t)$$
$$= -a_{21}p_1(D_2, t) - a_{22}p_2(D_2, t) - c_2(p_2(D_2, t)$$
$$+ (a_{11} + c_1)p_1(D_1 + D_2, t))$$
$$- (a_{11} + c_1)(a_{11}p_1(D_2 + D_1, t) + p_2(D_2, t))$$
$$= -a_{21}P_1(t - D_1) - a_{22}P_2(t) - c_2(P_2(t)$$
$$+ (a_{11} + c_1)P_1(t)) - (a_{11} + c_1)(a_{11}P_1(t) + P_2(t)), \tag{134}$$

where $P_1(t)$ and $P_2(t)$ are calculated using the integral representation (16)–(18). Note also that these integrals are computed using the trapezoidal rule.

Fig. 1 shows that the predictor controller exponentially stabilizes the system. The control signal reaches first $X_2(t)$, since the delay from the input to $X_2(t)$ is 0.4, which is smaller than the total delay from the input to $X_1(t)$. After 1.2 s, which is the total delay from the input to $X_1(t)$, the controller starts stabilizing $X_1(t)$. Then both $X_1(t)$ and $X_2(t)$ converge exponentially to zero (see Fig. 2).
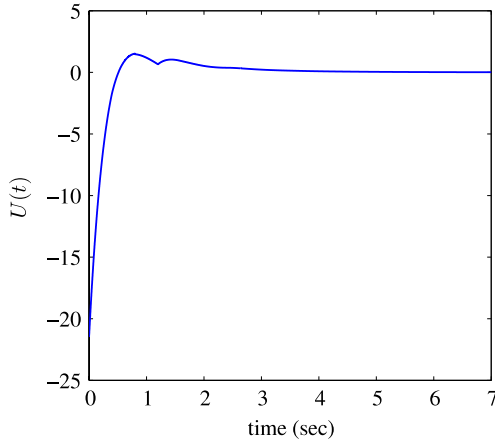
**Fig. 2.** Control effort for the simulation example.

## 5. Conclusions

We present a backstepping design for an exponentially unstable system with simultaneous input and state delay. Our design is predictor-based since it uses the predicted values of the states on given intervals. Using the boundness of the backstepping transformation and its inverse, we prove exponential stability of the closed-loop system using a properly weighted Lyapunov–Krasovskii functional.

A backstepping-like design for linear systems with only state delay is the one considered in Jankovic (2009a). The major difference with the design in Jankovic (2009a) and the one considered here, is that in Jankovic (2009a) delays are not allowed in the virtual inputs (which is the difficult case considered here). The present procedure can be modified to incorporate state delays that are in other positions other than the virtual inputs. In the case of a system with only input delay (irrespective of the form of the system, i.e., either if the system is a chain of integrators with input delay, e.g. Mazenc, Mondie, and Niculescu (2003), or a system in feedforward form, e.g. Jankovic (2010), etc.) the resulting control law is the predictor-based/finite spectrum assignment controller from Artstein (1982), Fiagbedzi and Pearson (1986), Krstic and Smyshlyaev (2008a) and Manitius and Olbrot (1979) with the gain $K$ being designed using the classical backstepping procedure for linear systems from Krstic et al. (1995). In the case of a system with simultaneous input and state delays a backstepping-like design comparable with the one considered here is the one in Jankovic (2010) for the special case of a chain of delayed integrators and input delay. In this special case the resulting control law from the present work turns out to be the same with the one in Jankovic (2010).

The present results can be also applied in the case where there are delays in other states too, and not just in the virtual inputs. Thus, the class of systems such that the present method can be applied is not limited. Considering the problem where the delays or the coefficients $a_{ij}$ are unknown, is a completely different and very challenging problem. Following the infinite dimensional backstepping technique, this problem has been solved for the case where there is only unknown input delay (Bresch-Pietri & Krstic, 2009a) and extended to the case of unknown input delay and plant parameters in Bresch-Pietri and Krstic (2009b). In Bekiaris-Liberis and Krstic (2010) a problem with unknown input and state delays is solved for a class of linear feedforward systems. Application of the design methods from Bekiaris-Liberis and Krstic (2010), Bresch-Pietri and Krstic (2009a,b) in the present case seems promising and can be pursued as a forthcoming research topic.

## Appendix

Here we give the proof of Lemma 5.

**Proof of Lemma 5.** By solving (41)–(43), and by taking into account that this ODE in $x$ system is in strict-feedback form, we get

$$p_1(x, t) = \int_{-D_1}^{x-D_1} v_{11}(x - y - D_1) p_2(y, t) dy$$
$$+ v_{11}(x) X_1(t), \quad x \in \left[0, \sum_{k=1}^{n} D_k\right] \tag{135}$$

$$p_2(x, t) = \sum_{i=1}^{2} \int_{-D_i}^{x-D_i} v_{2i}(x - y - D_i) p_{i+1}(y, t) dy$$
$$+ \sum_{i=1}^{2} v_{2i}(x) X_i(t), \quad x \in \left[0, \sum_{k=2}^{n} D_k\right] \tag{136}$$

$$\vdots$$

$$p_n(x, t) = \sum_{i=1}^{n-1} \int_{-D_i}^{x-D_i} v_{ni}(x - y - D_i) p_{i+1}(y, t) dy$$
$$+ \sum_{i=1}^{n} v_{ni}(x) X_i(t) + \int_{0}^{x} v_{nn}(x - y) u(y, t) dy,$$
$$x \in [0, D_n], \tag{137}$$

where

$$e^{A_0 x} = \begin{bmatrix} v_{11}(x) & 0 & 0 & \ldots & 0 \\ v_{21}(x) & v_{22}(x) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n1}(x) & v_{n2}(x) & \ldots & \ldots & v_{nn}(x) \end{bmatrix} \tag{138}$$

$$A_0 = \begin{bmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ a_{21} & a_{22} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & \ldots & a_{nn} \end{bmatrix}. \tag{139}$$

By applying Young's and Cauchy–Schwarz's inequalities to Eqs. (135)–(137) we get

$$p_1^2(x, t) \leq A_1 \left( X_1^2(t) + \int_{-D_1}^{x-D_1} p_2^2(y, t) dy \right) \tag{140}$$

$$p_2^2(x, t) \leq A_2 \left( X_1^2(t) + X_2^2(t) + \sum_{i=1}^{2} \int_{-D_i}^{x-D_i} p_{i+1}^2(y, t) dy \right) \tag{141}$$

$$\vdots$$

$$p_n^2(x, t) \leq A_n \left( \sum_{i=1}^{n} X_i^2(t) + \sum_{i=1}^{n-1} \int_{-D_i}^{x-D_i} p_{i+1}^2(y, t) dy + \int_{0}^{x} u^2(y, t) dy \right), \tag{142}$$

where, in each of the above bounds, $x \in [0, \sum_{k=i}^{n} D_k]$, respectively. Also

$$A_i = 2i \max \left[ \sup_{x \in \left[0, \sum_{k=i}^{n} D_k\right]} v_{i1}^2(x), \ldots, \sup_{x \in \left[0, \sum_{k=i}^{n} D_k\right]} v_{ii}^2(x), \right.$$

$$\sup_{x \in \left[0, \sum\limits_{k=i}^{n} D_k\right]} \int_{-D_1}^{x-D_1} v_{i1}(x-y-D_1)^2 \mathrm{d}y, \ldots,$$

$$\left. \sup_{x \in \left[0, \sum\limits_{k=i}^{n} D_k\right]} \int_{-D_i}^{x-D_i} v_{ii}(x-y-D_i)^2 \mathrm{d}y \right]. \tag{143}$$

If we take now into account that $p_i(x-D_{i-1}, t) = X_i(t+x-D_{i-1}) = \xi_i(x, t)$ we can rewrite (140)–(142) as

$$p_1^2(x, t) \le A_1 \left( X_1^2(t) + \|\xi_2(t)\|^2 + \int_0^{x-D_1} p_2^2(y, t) \mathrm{d}y \right) \tag{144}$$

$$p_2^2(x, t) \le A_2 \left( \sum_{k=1}^{2} X_i^2(t) + \sum_{i=1}^{2} \|\xi_{i+1}(t)\|^2 + \sum_{i=1}^{2} \int_0^{x-D_i} p_{i+1}^2(y, t) \mathrm{d}y \right) \tag{145}$$

$$\vdots$$

$$p_n^2(x, t) \le A_n \left( \sum_{k=1}^{n} X_i^2(t) + \sum_{i=1}^{n-1} \|\xi_{i+1}(t)\|^2 + \sum_{i=1}^{n-1} \int_0^{x-D_i} p_{i+1}^2(y, t) \mathrm{d}y + \int_0^{x} u^2(y, t) \mathrm{d}y \right), \tag{146}$$

where in each of the above relations, $x \in [0, \sum_{k=i}^{n} D_k]$, respectively. From the above equations, recursively, we can take the upper bound of the lemma. To see this, we start from relation (144) and observe that the boundness of $p_1^2(x, t)$ depends only on the boundness of $X_1(t)$ and $\xi_2(x, t)$ (thats is, $p_1^2(x, t)$ remains bounded for all $x \in [0, \sum_{k=1}^{n} D_k]$), if for all $x \in [0, \sum_{k=2}^{n} D_k]$, $p_2^2(x, t)$ is upper bounded. We proceed now by proving that the boundness of $p_2^2(x, t)$ depends only on the boundness of $X_1(t)$, $X_2(t)$, $\xi_2(x, t)$ and $\xi_3(x, t)$ (thats is, $p_2^2(x, t)$ remains bounded for all $x \in [0, \sum_{k=2}^{n} D_k]$), if for all $x \in [0, \sum_{k=3}^{n} D_k]$, $p_3^2(x, t)$ is upper bounded. From relation (145) (and by noting that $\int_0^{x-D_1} p_2^2(y, t) \mathrm{d}y \le \int_0^{x} p_2^2(y, t) \mathrm{d}y$ for all $x$ for which this equation holds, i.e., $\forall x \in [0, \sum_{k=2}^{n} D_k]$) by using the comparison principle and by exploiting the fact that $\mathrm{e}^{A_2 x} \le \mathrm{e}^{|A_2| \sum_{k=2}^{n} D_k}$, $\forall x \in [0, \sum_{k=2}^{n} D_k]$, we get that

$$\int_0^{x} p_2^2(y, t) \mathrm{d}y \le A_2 \mathrm{e}^{|A_2| \sum\limits_{k=2}^{n} D_k} \left( \sum_{k=2}^{n} D_k \left( \sum_{k=1}^{2} X_i^2(t) + \sum_{i=1}^{2} \|\xi_{i+1}(t)\|^2 \right) \int_0^{x} \int_0^{y-D_2} p_3^2(r, t) \mathrm{d}r \mathrm{d}y \right). \tag{147}$$

Plugging the above bound in to relation (145) we get a bound of $p_2^2(x, t)$ that depends on $p_3^2(x, t)$. Moreover, using the relation

$$p_3^2(x, t) \le A_3 \left( \sum_{k=1}^{3} X_i^2(t) + \sum_{i=1}^{3} \|\xi_{i+1}(t)\|^2 + \sum_{i=1}^{3} \int_0^{x-D_i} p_{i+1}^2(y, t) \mathrm{d}y \right), \tag{148}$$

and the previous bound, we get

$$p_3^2(x, t) \le A_3 \left( \sum_{k=1}^{3} X_i^2(t) + \sum_{i=1}^{2} \|\xi_{i+1}(t)\|^2 \right)$$

$$+ A_3 \left( A_2 \mathrm{e}^{|A_2| \sum\limits_{k=2}^{n} D_k} \sum_{k=2}^{n} D_k \left( \sum_{k=1}^{2} X_i^2(t) + \sum_{i=1}^{2} \|\xi_{i+1}(t)\|^2 \right) \right)$$

$$+ A_3 A_2 \mathrm{e}^{|A_2| \sum\limits_{k=2}^{n} D_k} \int_0^{x} \int_0^{y} p_3^2(r, t) \mathrm{d}r \mathrm{d}y + A_3 \int_0^{x} p_3^2(x, t) \mathrm{d}y$$

$$+ A_3 \int_0^{x-D_3} p_4^2(x, t), \quad x \in \left[0, \sum_{k=3}^{n} D_k\right]. \tag{149}$$

Note that the delayed terms in the integral for $p_3^2(x, t)$ can be removed since now $x \in [0, \sum_{k=3}^{n} D_k]$ which is the domain of definition for $p_3(x, t)$ (and of course this integral is larger than the delayed one). By changing the order of integration in the double integral of the previous relation, we can rewrite

$$\int_0^{x} \int_0^{y} p_3^2(r, t) \mathrm{d}r \mathrm{d}y = \int_0^{x} (x-y) p_3^2(y, t) \mathrm{d}y. \tag{150}$$

By observing that $\int_0^{x} (x-y) p_3^2(y, t) \mathrm{d}y \le \sum_{k=3}^{n} D_k \int_0^{x} p_3^2(y, t) \mathrm{d}y$, $\forall x \in \sum_{k=3}^{n} D_k$, and applying again the comparison principle for $\int_0^{x} p_3^2(y, t) \mathrm{d}y$ we can bound $p_3^2(x, t)$ from $p_4^2(x, t)$ and consequently also $p_2^2(x, t)$. Repeating this process until $p_n^2(x, t)$ (the boundness of which depends only on the boundness of $\|u(x, t)\|^2$), we derive the bound of the lemma. $\square$

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