



Lagrangian controls for traffic flow with autonomous and connected vehicles

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Monitoring vehicular traffic (2008)

MOBILE CENTURY – Using GPS Mobile Phones as Traffic Sensors



Daniel Work, Olli-Pekka Tossavainen, Alexandre Bayen
Systems Engineering, UC Berkeley

NOKIA
Connecting People

CITRIS 
California Center
for Innovative
Transportation

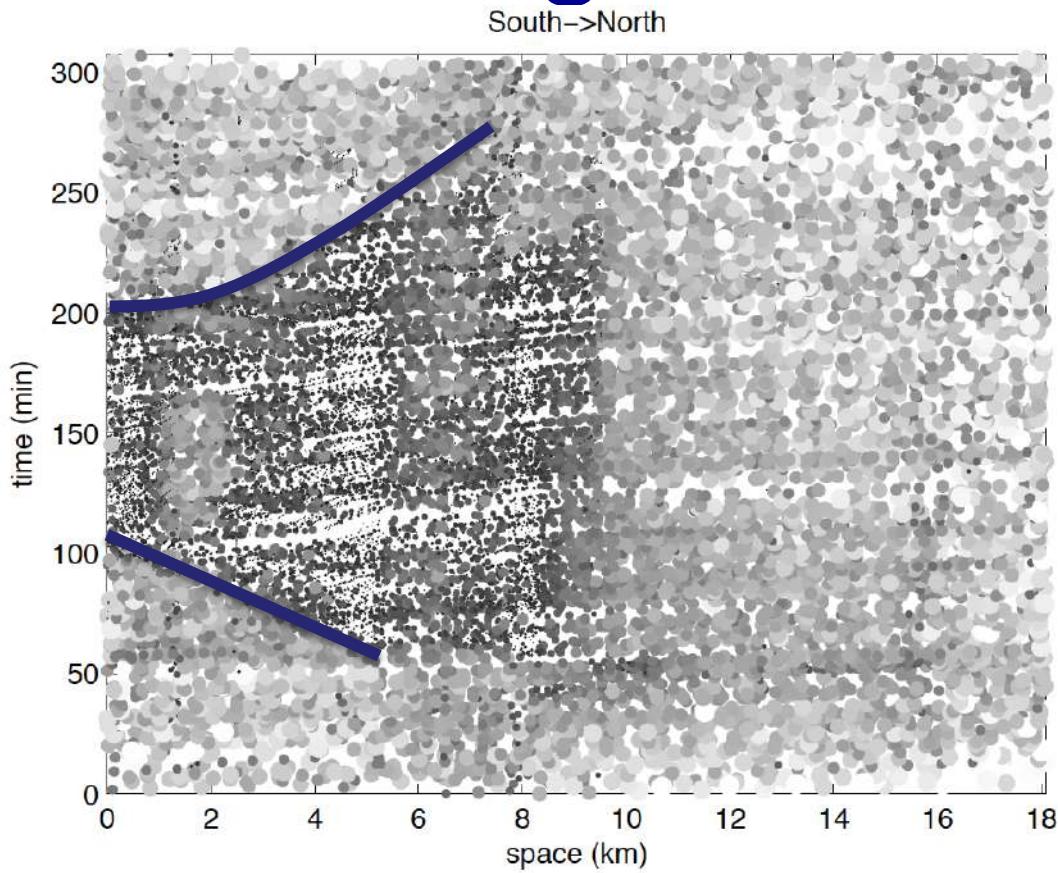
 Caltrans

 Institute of transportation studies

 TEKES



Monitoring vehicular traffic (2008)



Rankine-Hugoniot condition

$$\text{Speed} = \frac{\text{upstream flow} - \text{downstream flow}}{\text{upstream density} - \text{downstream density}}$$

2018:
Controlling traffic with CAVs?
(microscopic level)

Control of Newtonian systems

$$\begin{cases} \dot{x}_i &= v_i \\ \dot{v}_i &= \frac{1}{N} \sum_{j=1}^N a(\|x_j - x_i\|) (v_j - v_i) + u_i \end{cases}$$



$$\dot{v}_i = \frac{1}{N} \sum_{j=1}^N \frac{v_j - v_i}{(1 + \|x_j - x_i\|^2)^\beta}$$

$$\beta \leq \frac{1}{2}$$

the system will converge to a consensus

Control of Cucker-Smale: Caponigro, Fornasier, P. Rossi, Trelat
Cucker-Smale : consensus (flocking) conditions for $\beta > 1/2$
Control/consensus of large groups: Murray, Olfati-Saber, Egerstedt,
Ha-Tadmor: hydrodynamic limit of CS
Leonard, Krishnaprasad, Saberi, Bicchi, Kumar, Pappas, ...
Motsch-Tadmor: local interactions, asymmetric

Particle systems: Reynolds, Vicsek, Ben-Jacob et al, Krause, Couzin, Helbing, ...
Degond, Motsch, Carrillo, Fornasier, Toscani, Figalli, ...

Are we reinventing the wheel?

Control affine system: $\dot{x} = f(x) + \sum_i u_i g_i(x) = f(x) + G(x)u$
such that $\dot{x} = f(x)$ admits Lyapunov function, then
Jurdjevid-Quinn feedback is:

$$u(x) = -\alpha (\nabla V(x) \cdot G(x))^t.$$

Bacciotti-Ceragioli, ESAIM:COCV 1999.

Stability of $\dot{x} \in F(x)$ for V only continuous, F u.s.c. with nonempty compact convex values.

Set valued derivatives

$$\dot{V}^F(x) = \{a \in \mathbb{R} : \exists v \in F(x) \text{ s.t. } p \cdot v = a, p \in \partial V(x)\}$$

Stabilization to the set $Z_v = \{x : 0 \in \dot{V}^F(x)\}$ under suitable conditions with Jurdjevic-Quinn feedback.

Other results in:

A. Bacciotti and L. Rosier, Liapunov Functions and Stability in Control Theory, Springer, Second Edition 2006.

Sparse control of Newtonian systems

Definition 3. For every $M > 0$ and every $(x, v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, let $U(x, v)$ be defined as the set of solutions of the variational problem

$$\min \left(B(v, u) + \gamma(B(x, x)) \sum_{i=1}^N \|u_i\| \right) \quad \text{subject to } \sum_{i=1}^N \|u_i\| \leq M, \quad (11)$$

where

$$\gamma(X) = \int_{\sqrt{X}}^{\infty} a(\sqrt{2N}r)dr. \quad (12)$$

then

$$u_i^\circ = -M \frac{v_{\perp i}}{\|v_{\perp i}\|}, \quad \text{and} \quad u_j^\circ = 0 \quad \text{for every } j \neq i.$$

Theorem 3. Fix $M > 0$ and consider the control u° law given by Definition 4. Then for every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ there exists $\tau_0 > 0$ small enough, such that for all $\tau \in (0, \tau_0]$ the sampling solution of (9) associated with the control u° , the sampling time τ , and initial datum (x_0, v_0) reaches the consensus region in finite time.

Proposition 3. The feedback control $u^\circ(t) = u^\circ(x(t), v(t))$ of Definition 4, associated with the solution $((x(t), v(t))$ of Theorem 2, is a minimizer of

$$\mathcal{R}(t, u) = \frac{d}{dt} V(t),$$

over all possible feedback controls in $U(x(t), v(t))$. In other words, the feedback control $u^\circ(t)$ is the best choice in terms of the rate of convergence to consensus.

Can we really control?

Traffic Jam without Bottleneck

Experimental evidence
for the physical mechanism of forming a jam

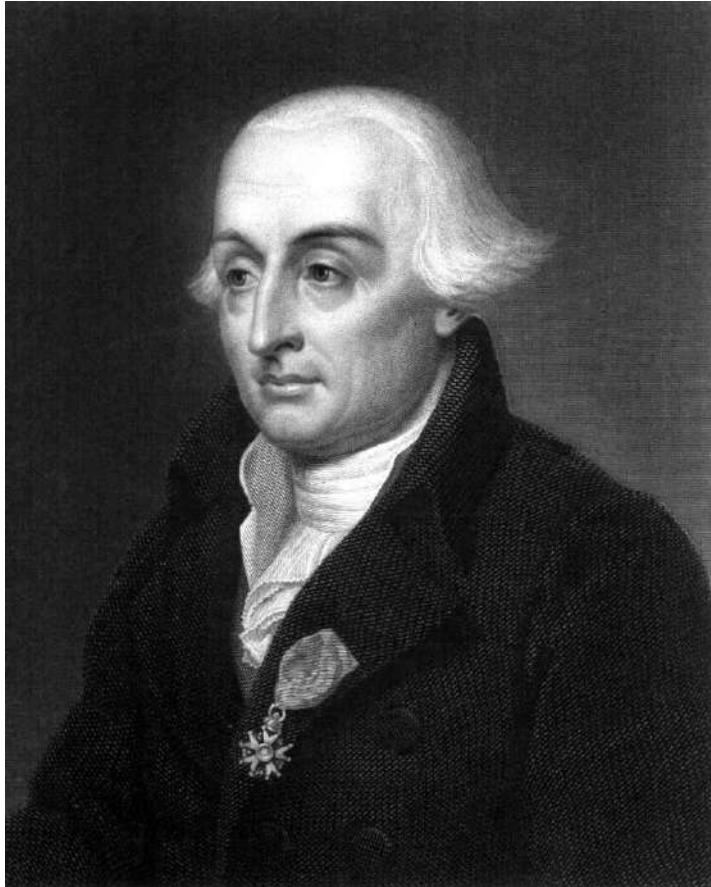
Yuki Sugiyama, Minoru Fukui, Macoto Kikuchi,
Katsuya Hasebe, Akihiro Nakayama, Katsuhiro Nishinari,
Shin-ichi Tadaki and Satoshi Yukawa

Movie 1

The Mathematical Society of Traffic Flow

2018:
Controlling traffic with CAVs?
(multilane multiscale level)

Lagrangian controls?



Giuseppe Lodovico Lagrangia
(Torino 1736 – Paris 1803)

From baptism registry in Sant Philippe Church
in Turin:

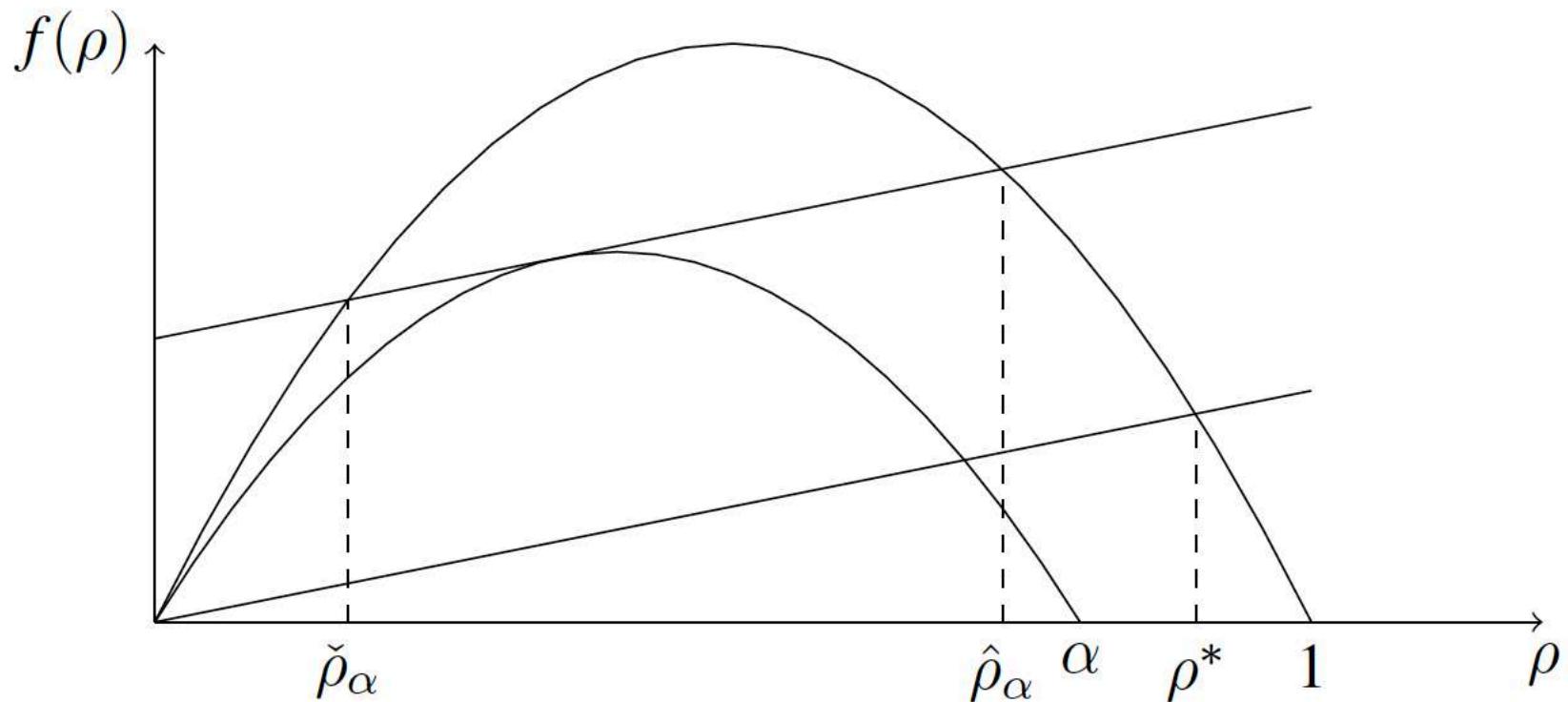
*Lagrangia Giuseppe Lodovico,
figlio del signor Giuseppe Francesco Lodovico e
di Teresa Grosso Lagrangia, nato il venticinque
gennaio dell'anno millesettecentotrentasei,
fu battezzato il 30 gennaio seguente.
Padrino fu il sig. Carlo Lagrangia e
Madrina l'ill. ma contessa
Anna Caterina Rebuffi di Traves.*

Delle Monache-Goatin model

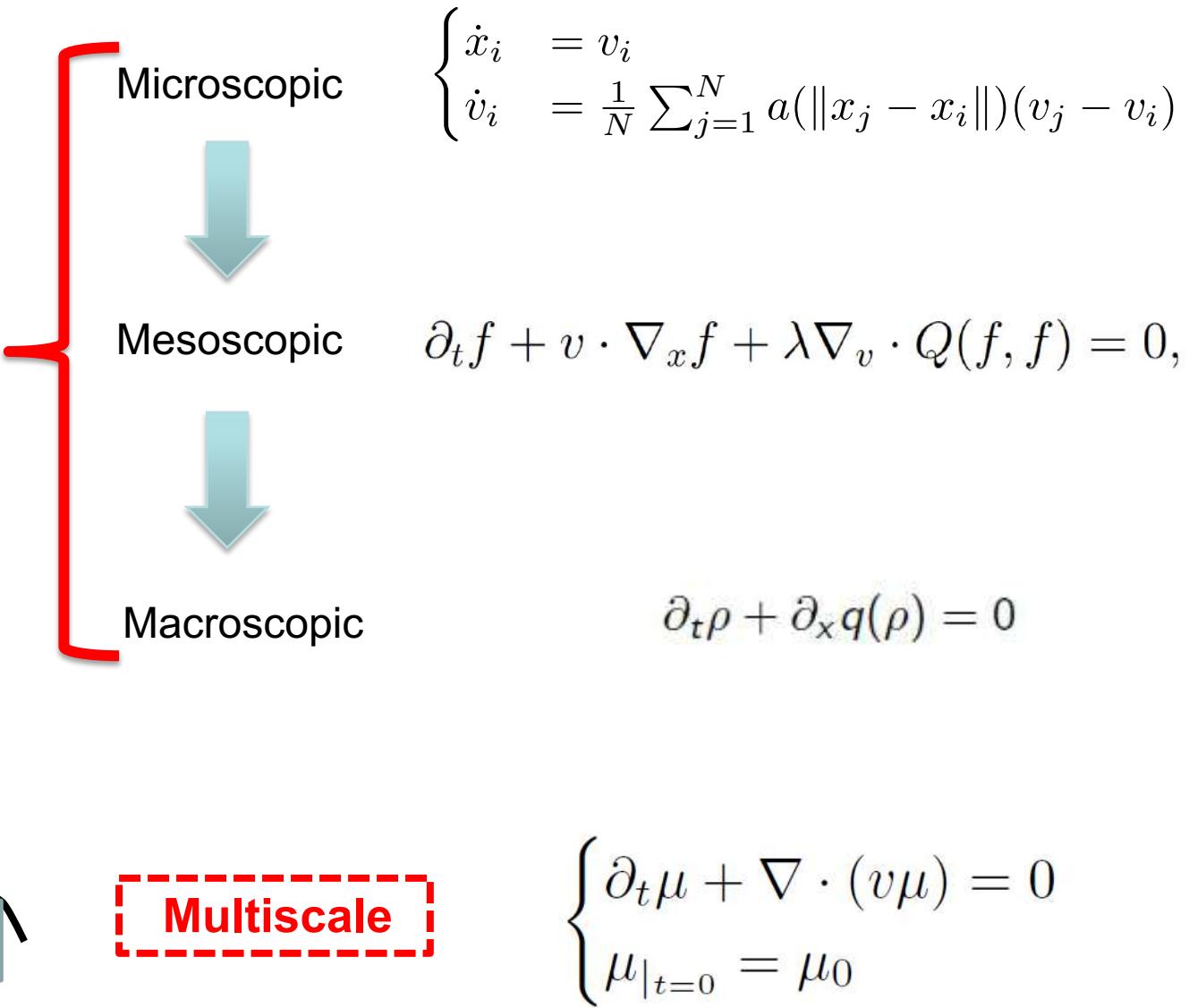
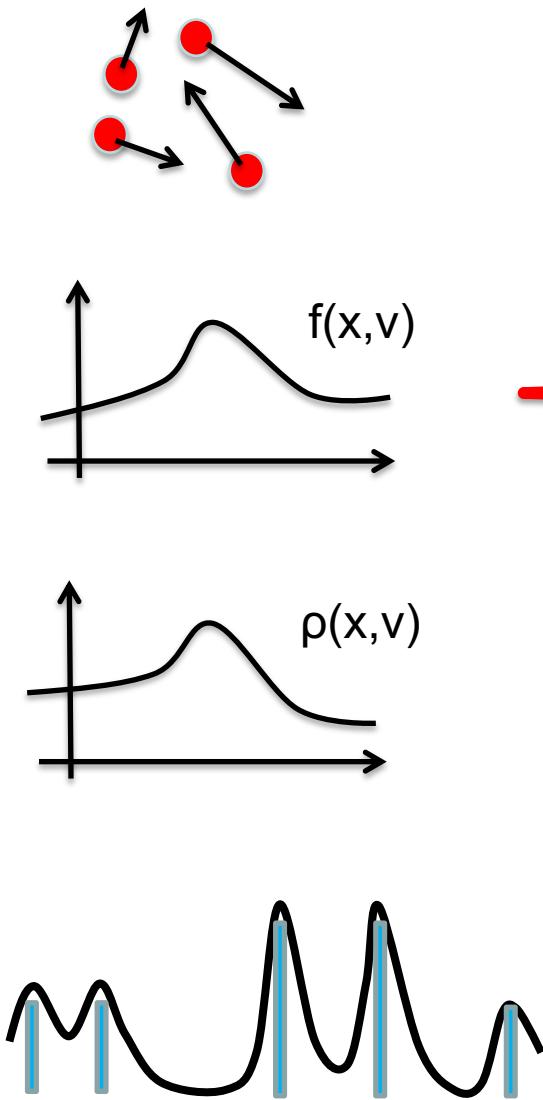
$$\partial_t \rho + \partial_x(\rho(1 - \rho)) = 0 \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}$$

$$f(\rho(t, y(t))) - \dot{y}(t)\rho(t, y(t)) \leq F_\alpha := \frac{\alpha}{4}(1 - \dot{y}(t))^2 \quad t \in \mathbb{R}^+$$

$$\dot{y}(t) = \omega(\rho(t, y(t)+)) \quad t \in \mathbb{R}^+$$



Modeling at different scales



Can we really control? (2)

Mean-field limit: Vlasov-Poisson equations

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = H \star \mu_N(x_i, v_i), \quad i = 1, \dots, N, \end{cases}$$

$$H(x, v) := a(|x|)v,$$



$$\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \star \mu) \mu]$$

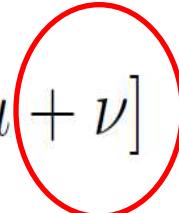
$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{(x_i(t), v_i(t))}$$

Propagation of chaos (Kac)
Dobrushin, ...

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = H \star \mu_N(x_i, v_i) + u_i, \quad i = 1, \dots, N, \end{cases}$$



$$\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \star \mu) \mu + \nu]$$



Control by leaders (with Fornasier and Rossi)

$$\begin{cases} \dot{y}_k = w_k, \\ \dot{w}_k = H \star \mu_N(y_k, w_k) + H \star \mu_m(y_k, w_k) + u_k & k = 1, \dots, m, \\ \dot{x}_i = v_i, \\ \dot{v}_i = H \star \mu_N(x_i, v_i) + H \star \mu_m(x_i, v_i) & i = 1, \dots, N, \end{cases} \quad (1)$$

$$\begin{cases} \dot{y}_k = w_k, \\ \dot{w}_k = H \star (\mu + \mu_m)(y_k, w_k) + u_k, & k = 1, \dots, m, \\ \partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \star (\mu + \mu_m)) \mu], \end{cases} \quad (2)$$

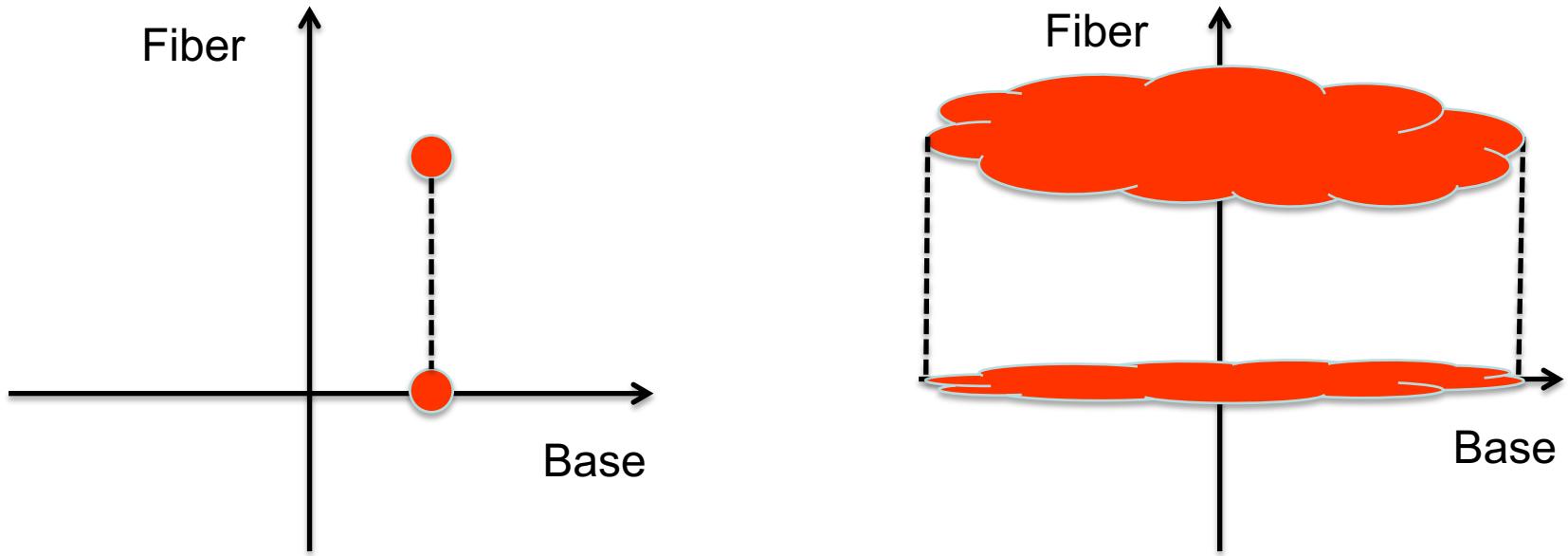
$$F_N(u) = \int_0^T \left\{ L(y_N(t), w_N(t), \mu_N(t)) + \frac{1}{m} \sum_{k=1}^m |u_k(t)| \right\} dt,$$

$$F(u) = \int_0^T \left\{ L(y(t), w(t), \mu(t)) + \frac{1}{m} \sum_{k=1}^m |u_k(t)| \right\} dt.$$

Theorem

F_N Γ -converges to F , thus optimal controls of (1) weakly converge to optimal controls of (2).

Measure differential equations



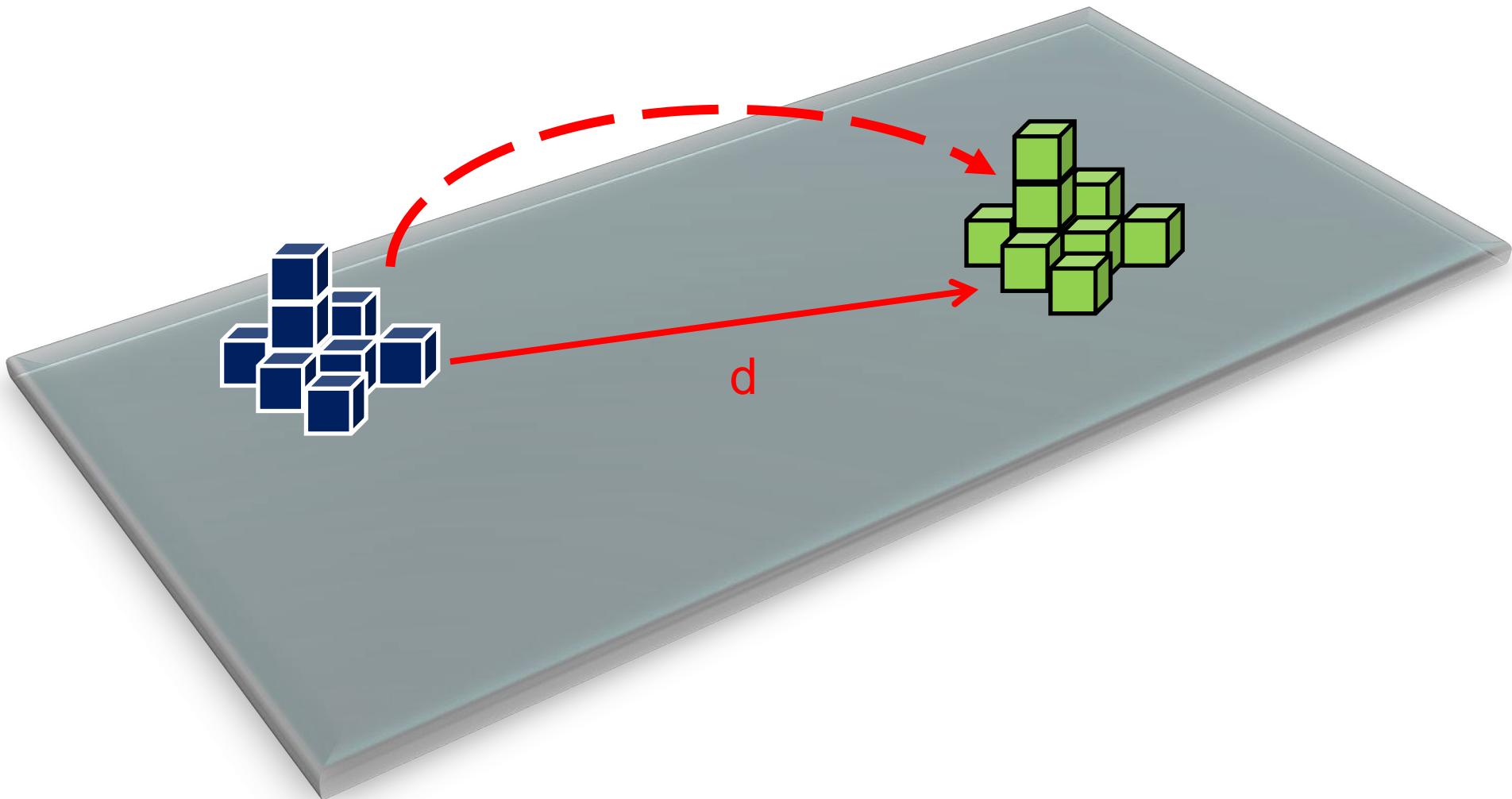
Definition

A Probability Vector Field (briefly PVF) on $\mathcal{P}(\mathbb{R}^n)$ is a map $V : \mathcal{P}(\mathbb{R}^n) \rightarrow \mathcal{P}(T\mathbb{R}^n)$ such that $\pi_1 \# V[\mu] = \mu$.

For every $f \in \mathbb{C}_c^\infty(\mathbb{R}^n)$,

$$\frac{d}{dt} \int_{\mathbb{R}^n} f(x) d\mu(t)(x) = \int_{T\mathbb{R}^n} (\nabla f(x) \cdot v) dV[\mu(t)](x, v). \quad (1)$$

Wasserstein (Veserstein) metric



Wasserstein metric (2)

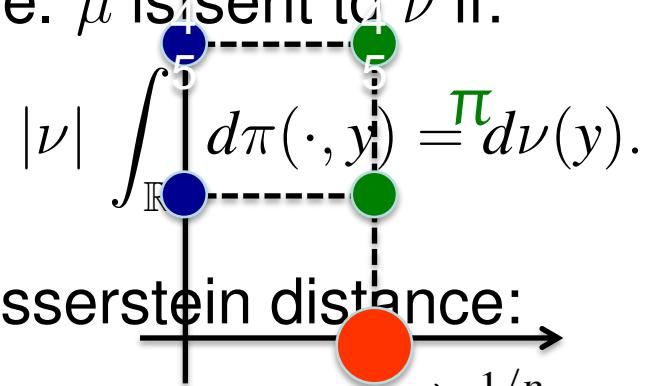
Optimal transport first proposed by Monge in 1781.

Particular case is given by the cost $c(x, y) = |x - y|^p$ with $p \geq 1$, defining the Wasserstein distance:

$$W_p(\mu, \nu) = \inf_{\gamma \# \mu = \nu} \left(\int_{\mathbb{R}^n} |\gamma(x) - x|^p d\mu \right)^{1/p}.$$

A probability measure π on $\mathbb{R}^d \times \mathbb{R}^d$, one can interpret it as a method to transfer a measure. μ is sent to ν if:

$$|\mu| \int_{\mathbb{R}^d} d\pi(x, \cdot)^\textcolor{red}{\mu} = d\mu(x),$$


$$|\nu| \int_{\mathbb{R}} d\pi(\cdot, y)^\textcolor{green}{\nu} = d\nu(y).$$


Monge-Kantorovich problem. Wasserstein distance:

$$W_p(\mu, \nu) = \left(|\mu| \min_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y) \right)^{1/p}.$$

where $\Pi(\mu, \nu)$ is the set of transference plans from μ to ν .

Measure differential equations (2)

Assumptions for existence of solutions:

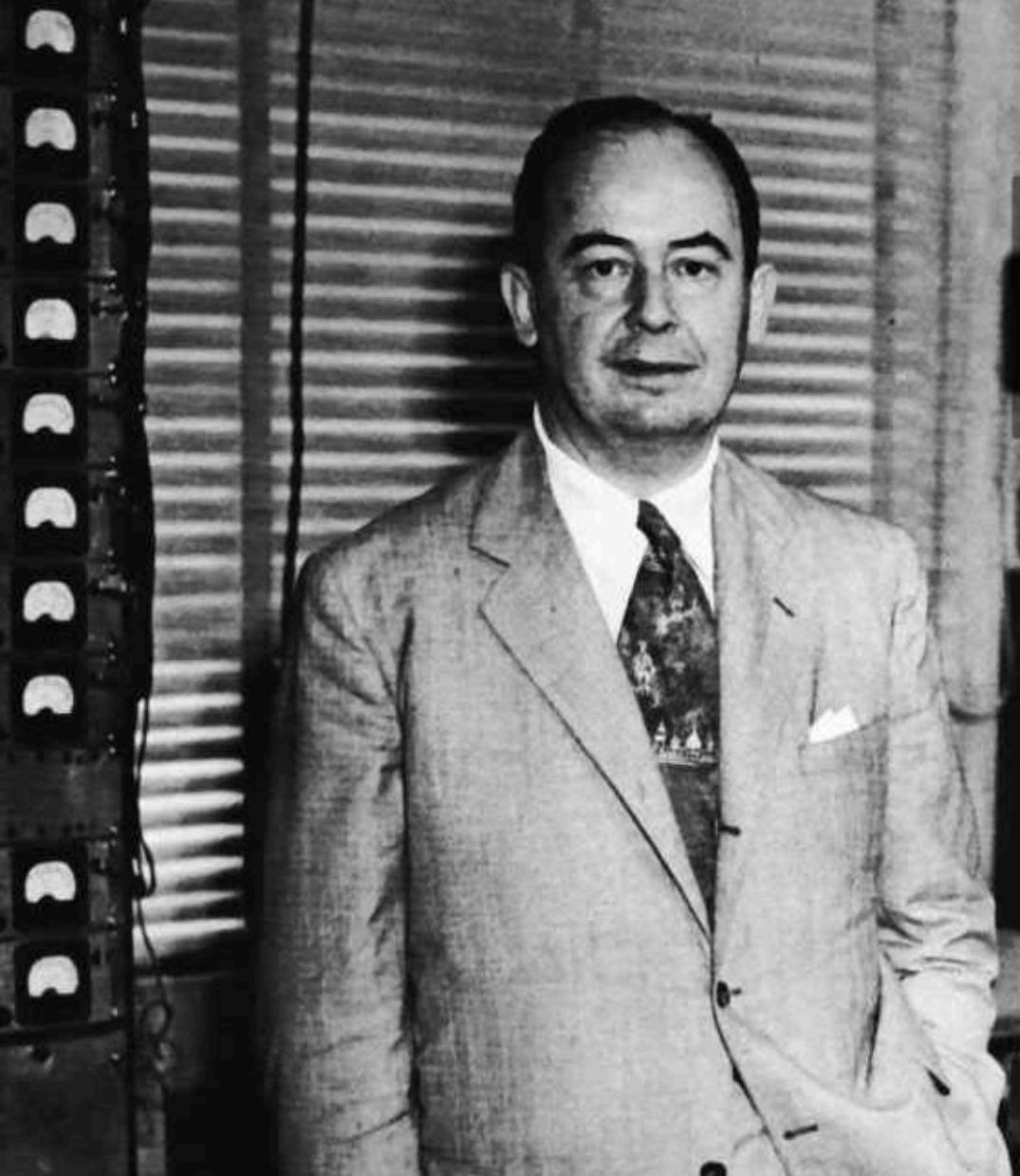
(H1) V is support sublinear, i.e. $\exists C > 0$ such that

$$\sup_{(x,v) \in \text{Supp}(V[\mu])} |v| \leq C \left(1 + \sup_{x \in \text{Supp}(\mu)} |x| \right).$$

(H2) The map $V : \mathcal{P}_c(\mathbb{R}^n) \rightarrow \mathcal{P}_c(T\mathbb{R}^n)$ is continuous (for the Wasserstein metrics $W^{\mathbb{R}^n}$ and $W^{T\mathbb{R}^n}$.)

Lipschitz semigroup using:

$$\begin{aligned} \mathcal{W}(V_1, V_2) &= \inf \left\{ \int_{T\mathbb{R}^n \times T\mathbb{R}^n} |v - w| \, dT(x, v, y, w) : \right. \\ &\quad \left. T \in P(V_1, V_2), \pi_{13}\#T \in P^{opt}(\mu_1, \mu_2) \right\} \end{aligned}$$



If people do not believe
that mathematics is simple,
it is only because they do not realize
how complicated life is.

John von Neumann

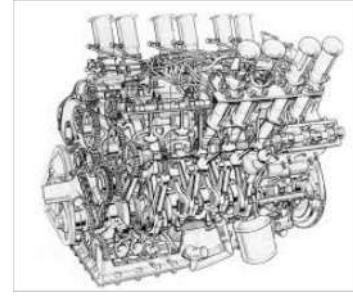
High complexity of traffic



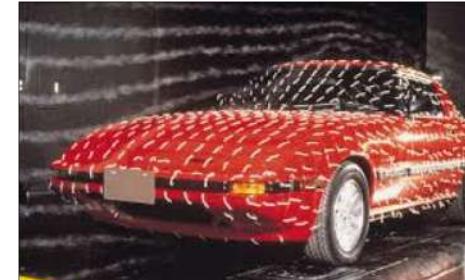
Infrastructure



Internal
dynamics



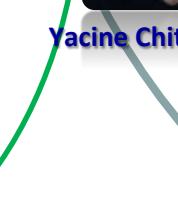
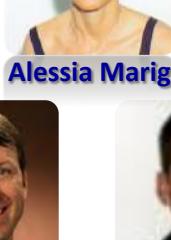
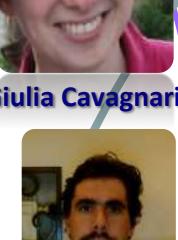
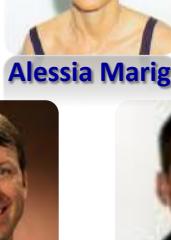
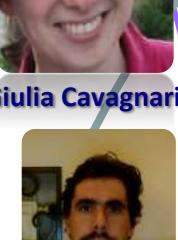
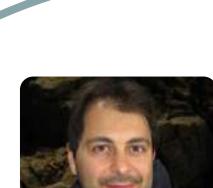
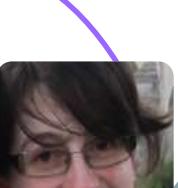
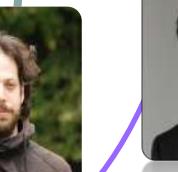
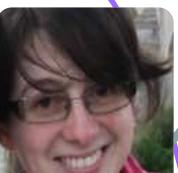
Agents
dynamics



CROWD DYNAMICS

VEHICULAR TRAFFIC

SOCIAL



ANIMAL GROUPS

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