Traffic flow control via PDEs: Delay-compensating and coordinated designs

Nikos Bekiaris-Liberis

Technical University of Crete, Greece

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Acknowledgements

PADECOT

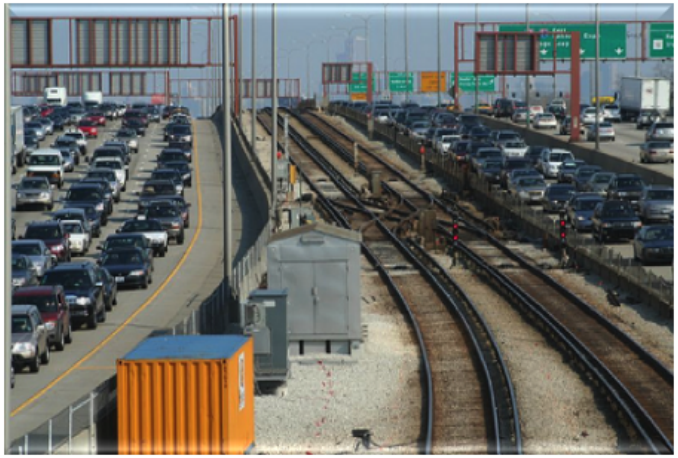
PArtial Differential Equation model-based COntrol of Traffic flow

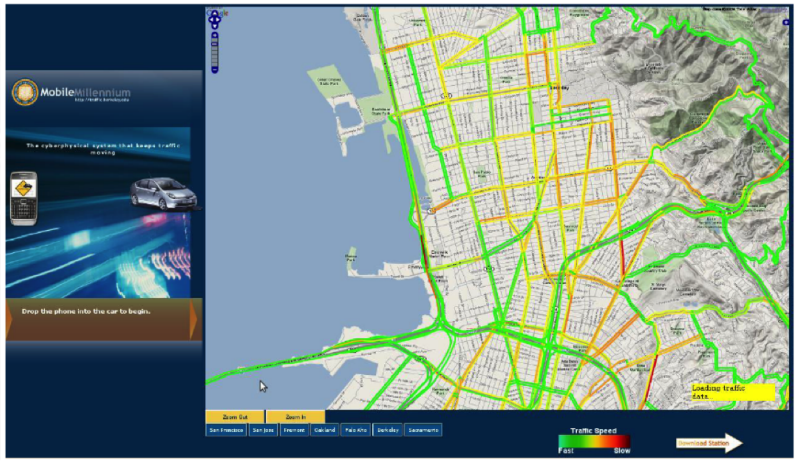
Marie Sklodowska-Curie grant agreement No 747898

Motivation

Real-time traffic control **reduces** costs/**improves** quality of daily transport via

i. Traffic congestion mitigation and travel times minimization
ii. Fuel consumption and emissions decrease
iii. Safety and comfort improvement





Traffic evolves/may be described in **continuous** time and space.

Develop control/estimation designs for **PDEs & PDEs-ODEs** from traffic:

- Explicit (e.g., appealing to implementation)
- Boundary actuation/sensing (e.g., minimum actuator/sensor requirements)
- Efficient use of available actuators/sensors (e.g., fault-tolerant)
- With stability guarantees (e.g., quantify performance)

Outline

- Motivation
- ACC design
- Predictor feedback
- Traffic flow control at distant bottlenecks
  - Quasilinear transport PDE-ODE interconnection
- Coordinated traffic flow control
  - Viscous Hamilton-Jacobi PDEs
- Current & future steps

ACC design

(w/ Claudio Roncoli & Markos Papageorgiou)

Motivation

- Delays in traffic with ACC/CACC vehicles
  - Actuator delay (engine response, throttle actuator, brake actuator, computational delay, ...)
  - Sensor delay (radar or lidar filter, wireless communication, sampling, wheel speed sensor, ...)

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- Negative effects
  - Decrease in capacity
  - String instability (comfort, safety, fuel consumption, tracking performance ...)
  - Individual vehicle instability (comfort, safety, fuel consumption, tracking performance ...)

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- Delays in traffic with ACC/CACC vehicles
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  - Sensor delay (radar or lidar filter, wireless communication, sampling, wheel speed sensor, ...)
- Negative effects
  - Decrease in capacity
  - String instability (comfort, safety, fuel consumption, tracking performance ...)
  - Individual vehicle instability (comfort, safety, fuel consumption, tracking performance ...)
- Goal: Compensate the delay to
  - Improve throughput
  - Improve string stability
  - Improve vehicle's stability

Measurements & Parameters

- relative spacing s_i
  vehicle's speed v_i
  vehicle's control variable U_i (e.g., desired acceleration).

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  vehicle's speed v_i
  vehicle's control variable U_i (e.g., desired acceleration).
- Combined delay D.

Vehicle Model

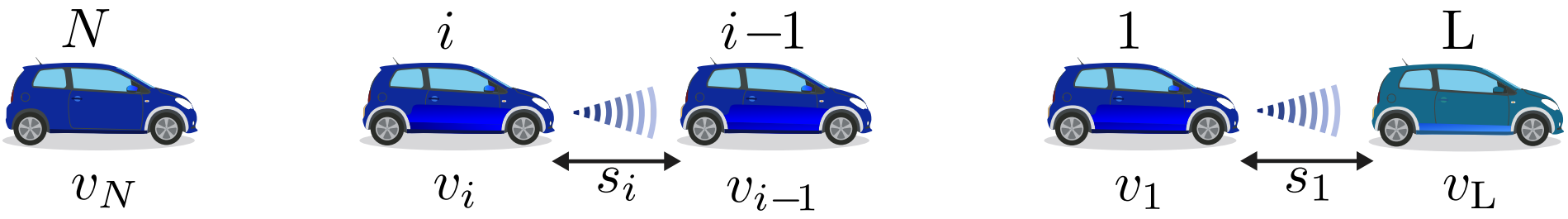

Figure: Homogenous platoon of vehicles in a single lane.

Vehicle's i dynamics

$$\begin{aligned} \dot{s}_i(t) &= v_{i-1}(t) - v_i(t) \\ \dot{v}_i(t) &= U_i(t - D), \end{aligned}$$

where

$$s_i = x_{i-1} - x_i - l,$$

x_j position
l length

Simulation (Set-up)

$D = 0.4$, $h = \frac{2}{\pi}$

$U_i(t) = \frac{\alpha}{h} s_i(t) - \alpha v_i(t) + b\left(v_{i-1}(t) - v_i(t)\right)$, $a = 1$, $b = 0.8$.

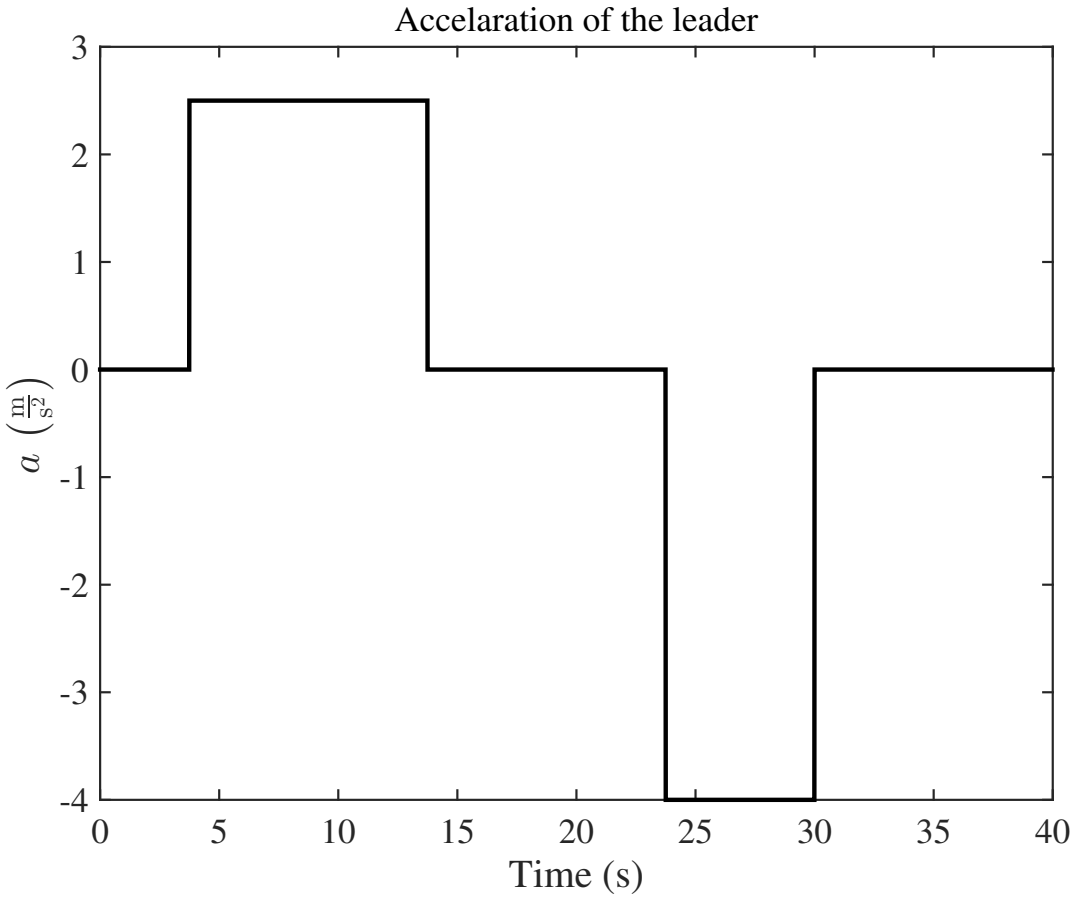

Figure: Acceleration maneuver of the leader with $\ddot{y}_{\mathrm{L}}(t) = a_{\mathrm{L}}(t)$.

Simulation (Uncompensated)

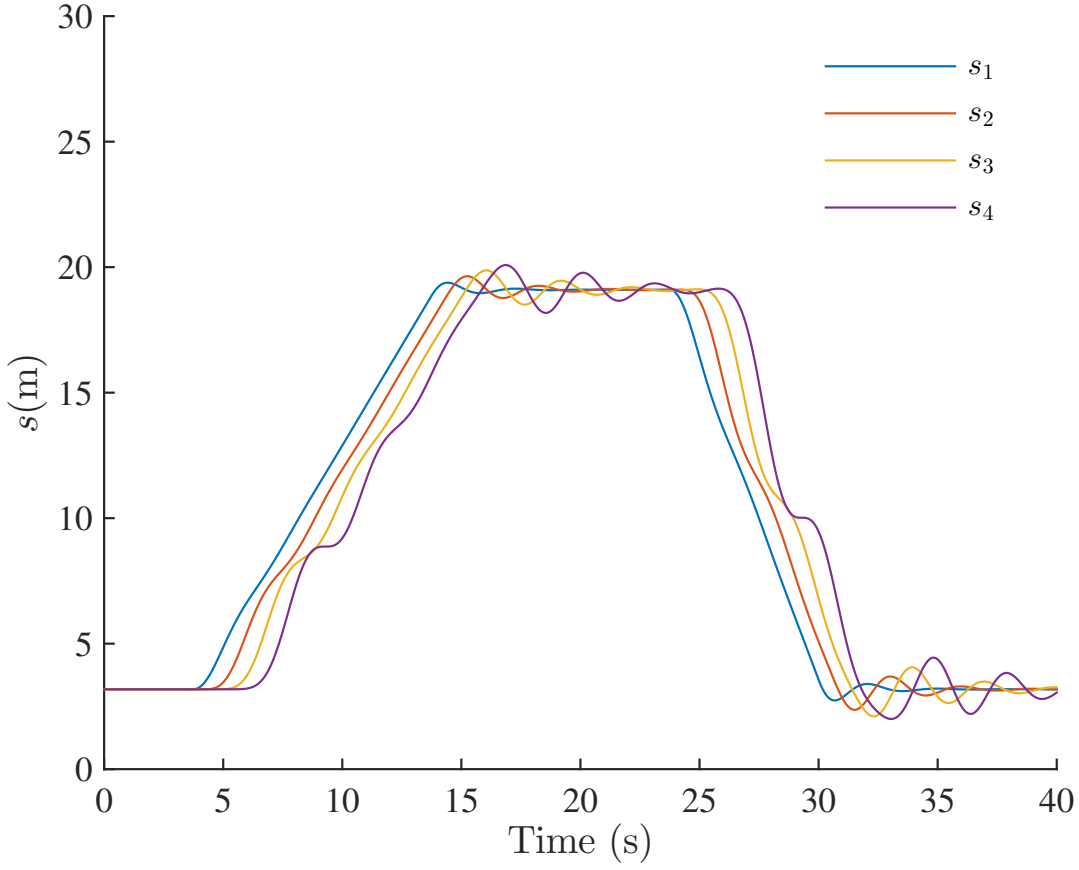

Figure: Spacing of four vehicles following a leader under the uncompensated ACC strategy.

Simulation (Uncompensated)

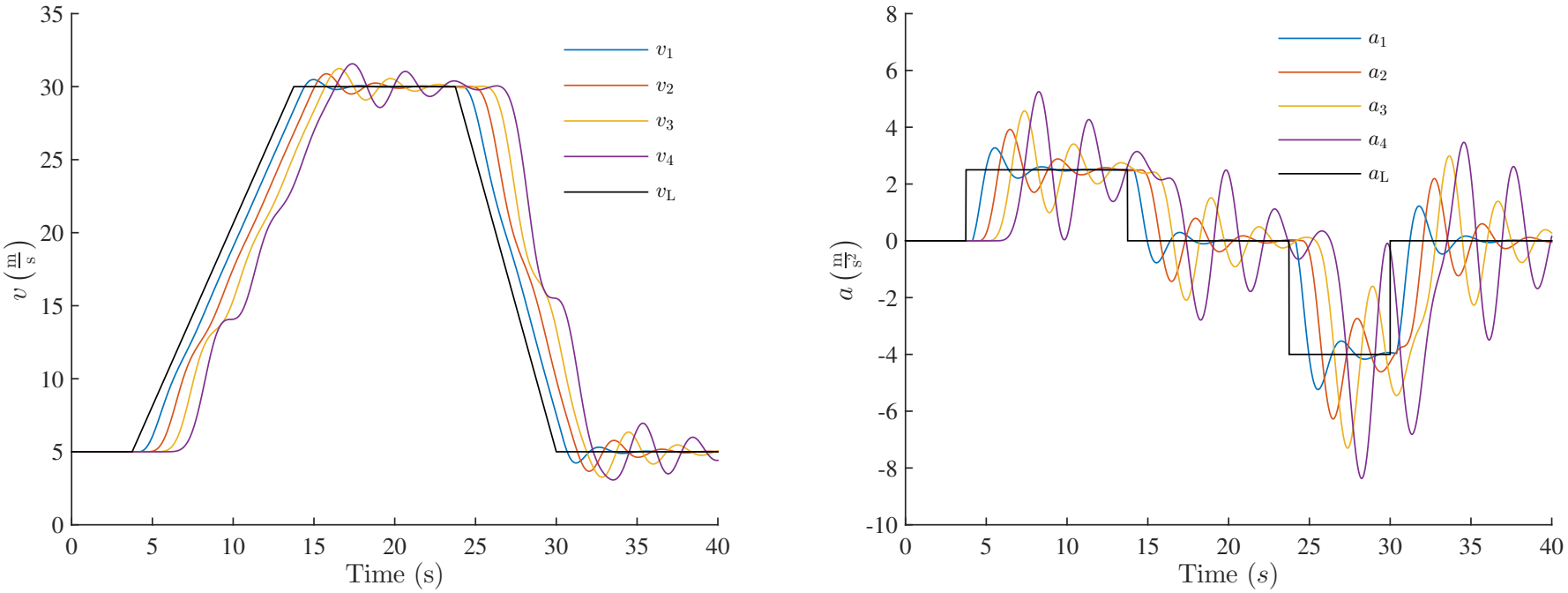

Figure: Speed (left) and acceleration (right) of four vehicles following a leader under the uncompensated ACC strategy.

Simulation (Predictor-Based)

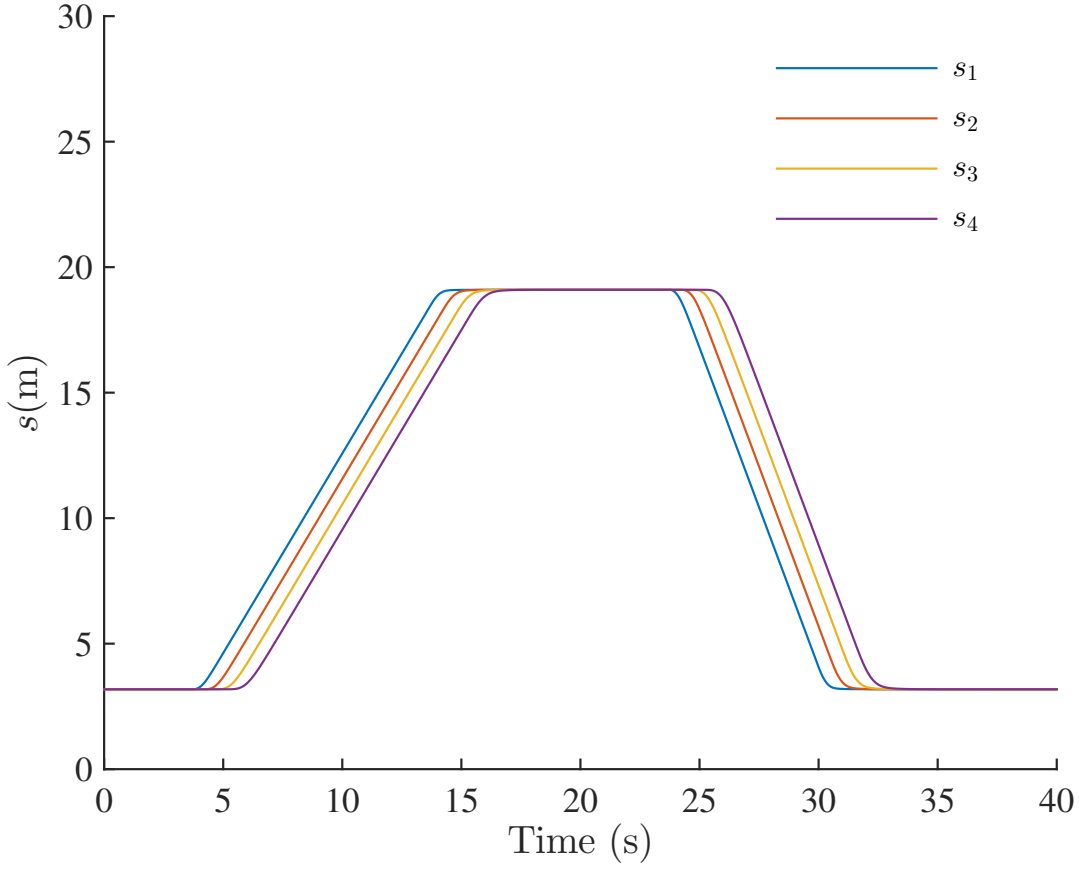

Figure: Spacing of four vehicles following a leader under the predictor-based ACC strategy.

Simulation (Predictor-Based)

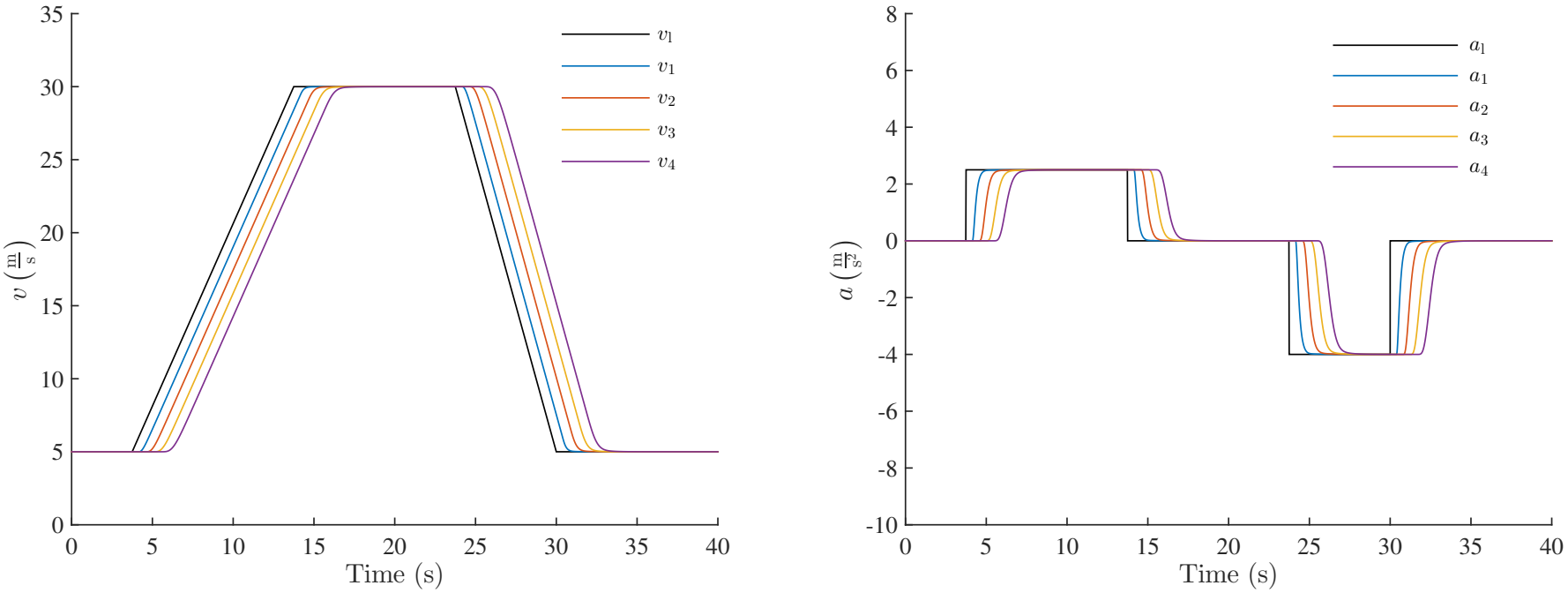

Figure: Speed (left) and acceleration (right) of four vehicles following a leader under the predictor-based ACC strategy.

Simulation (Comparison)

Table: Performance indices.

Performance index	Percentage improvement
J_{fuel}	28
$J_{\text{comfort},1}$	90
$J_{\text{comfort},2}$	20
$J_{\text{comfort},3}$	66
J_{safety}	53
$J_{\text{tracking},1}$	83
$J_{\text{tracking},2}$	51

Predictor Feedback

LTI Systems with Constant Delay

$$\dot{X}(t) = AX(t) + BU\left(t - D\right)$$

A - possibly unstable; D - arbitrarily large
Assume: (A, B) controllable and matrix K found such that $A + BK$ is Hurwitz.

LTI Systems with Constant Delay

$$\dot{X}(t) = AX(t) + BU(t-D)$$

Predictor feedback

$$U(t) = K \underbrace{\left[e^{AD} X(t) + \int_{t-D}^{t} e^{A(t-\theta)} BU(\theta) d\theta \right]}_{X(t+D)=P(t)}$$

Predictor-Based ACC Design

$$U_i(t) = K\left(e^{\Gamma D} X_i(t) + \int_{t-D}^{t} e^{\Gamma(t-\theta)} B U_i(\theta) d\theta\right),$$

with

$$\Gamma = \begin{bmatrix} 0 & 0 & -1 \\ \frac{1}{h} & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$X_i = \begin{bmatrix} s_i \\ \sigma_i \\ v_i \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where

$$\dot{\sigma}_i(t) = \frac{1}{h} s_i(t) - v_i(t), \quad i = 1, \ldots, N$$

Theorem & Implications

$\exists\ K$ *s.t. each individual vehicle is stable* $\forall\ D > 0$ *, and the platoon is* $p \in [1, \infty]$ *string stable* $\forall\ D < h$ *.*

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$\exists\ K$ s.t. each individual vehicle is stable $\forall\ D > 0$, and the platoon is $p \in [1, \infty]$ string stable $\forall\ D < h$.

Implications:

- stability
- zero steady-state spacing error
- string stability
- non-negative impulse response

Nonlinear Predictor Feedback

Nonlinear Systems with Constant Delay

$$\dot{X}(t) = f\left(X(t), U(t-D)\right)$$

Assumptions:

$$\begin{aligned} \dot{X} &= f\left(X, \kappa\left(X\right)\right), \quad \text{g.a.s.} \\ \dot{X} &= f\left(X, U\right), \quad \text{forward complete} \end{aligned}$$

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Predictor-feedback controller (predictor given implicitly in general):

$$\begin{aligned} U(t) &= \kappa\left(P(t)\right) \\ P(\theta) &= X(t) + \int_{t-D}^{\theta} f\left(P(\tau), U(\tau)\right) d\tau, \quad t - D \leq \theta \leq t \end{aligned}$$

Traffic Flow Control at Distant Bottlenecks

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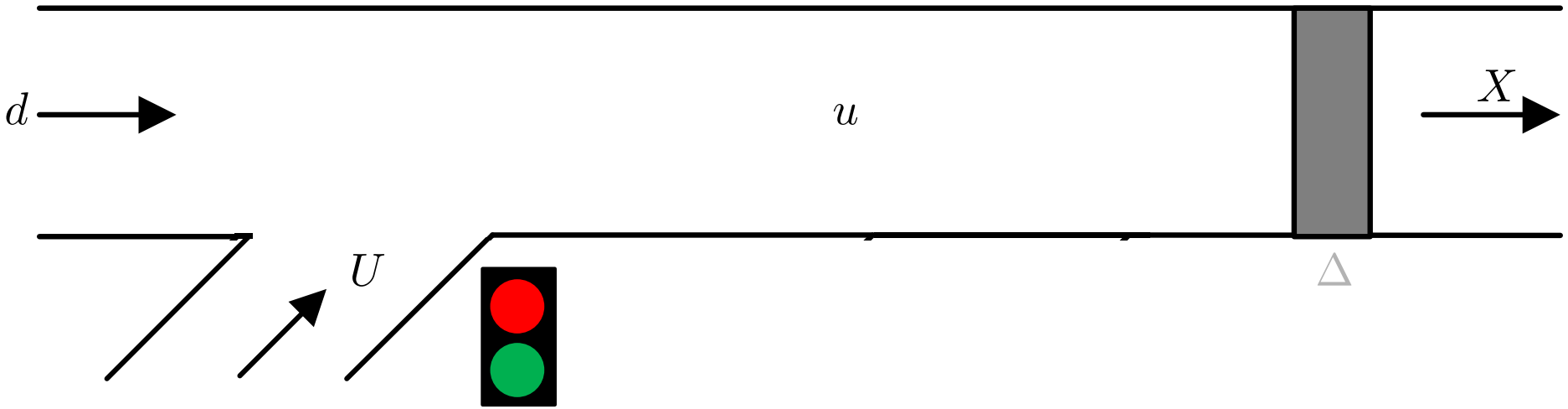

Figure: A highway stretch with one actuated on-ramp and a distant bottleneck.

Challenge: Long distance (delay) until vehicles (control input) reach the bottleneck area (plant)
Goal: Compensate delay and regulate the bottleneck flow

Traffic Flow Control at Distant Bottlenecks

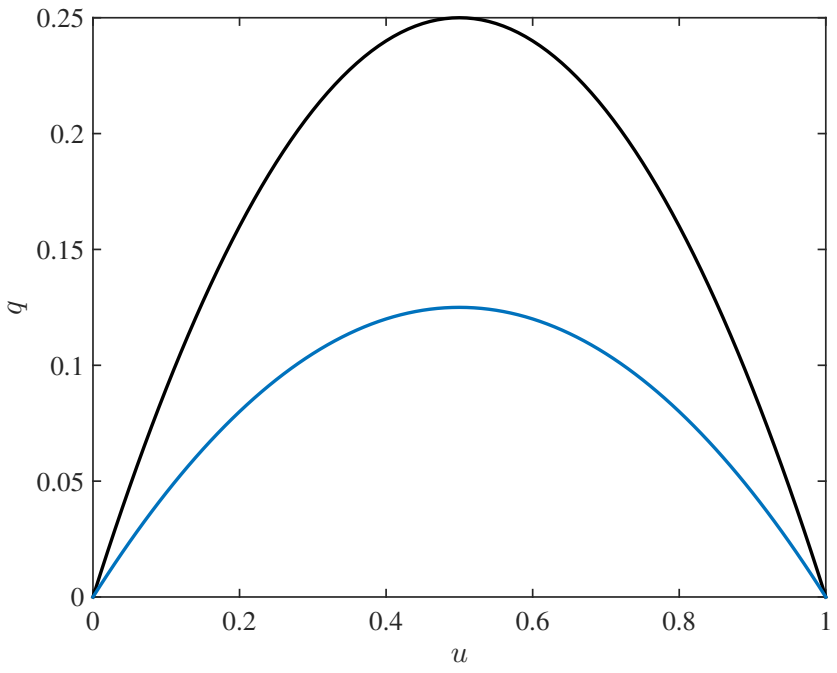

Figure: Fundamental diagrams of mainstream (black) and bottleneck areas (blue).

Traffic Flow Control at Distant Bottlenecks

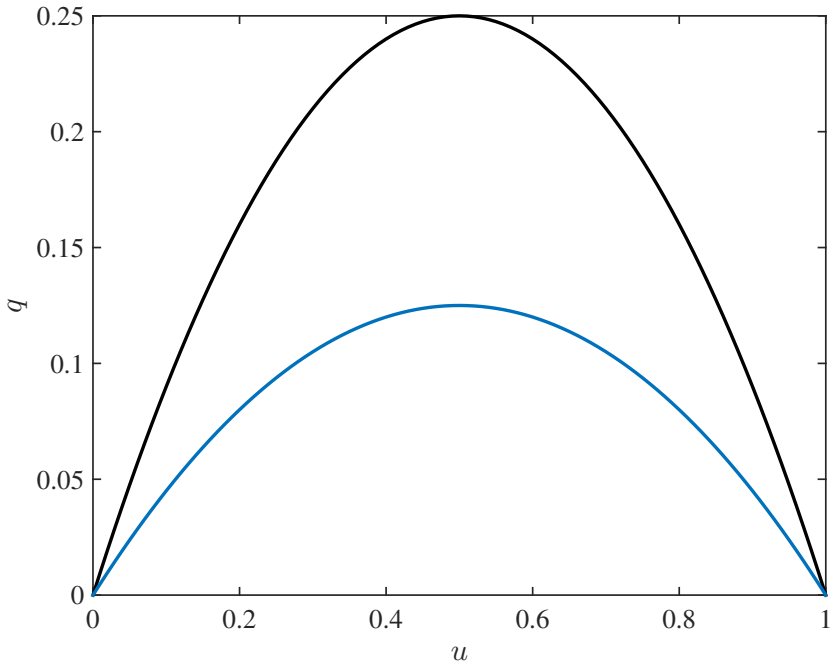

Figure: Fundamental diagrams of mainstream (black) and bottleneck areas (blue).

Model

$$
\begin{aligned}
\dot{X}(t) &= \frac{1}{\Delta}\left(u(0,t)\,(1-u(0,t)) - \frac{1}{2}X(t)\,(1-X(t))\right) \\
u_t(x,t) &= (1-2u(x,t))\,u_x(x,t) \\
q\,(u(1,t)) &= d(t) + r(t).
\end{aligned}
$$

Simulation of the Traffic Flow Control Model

Goal: Regulate bottleneck density to an uncongested equilibrium, say X^*.

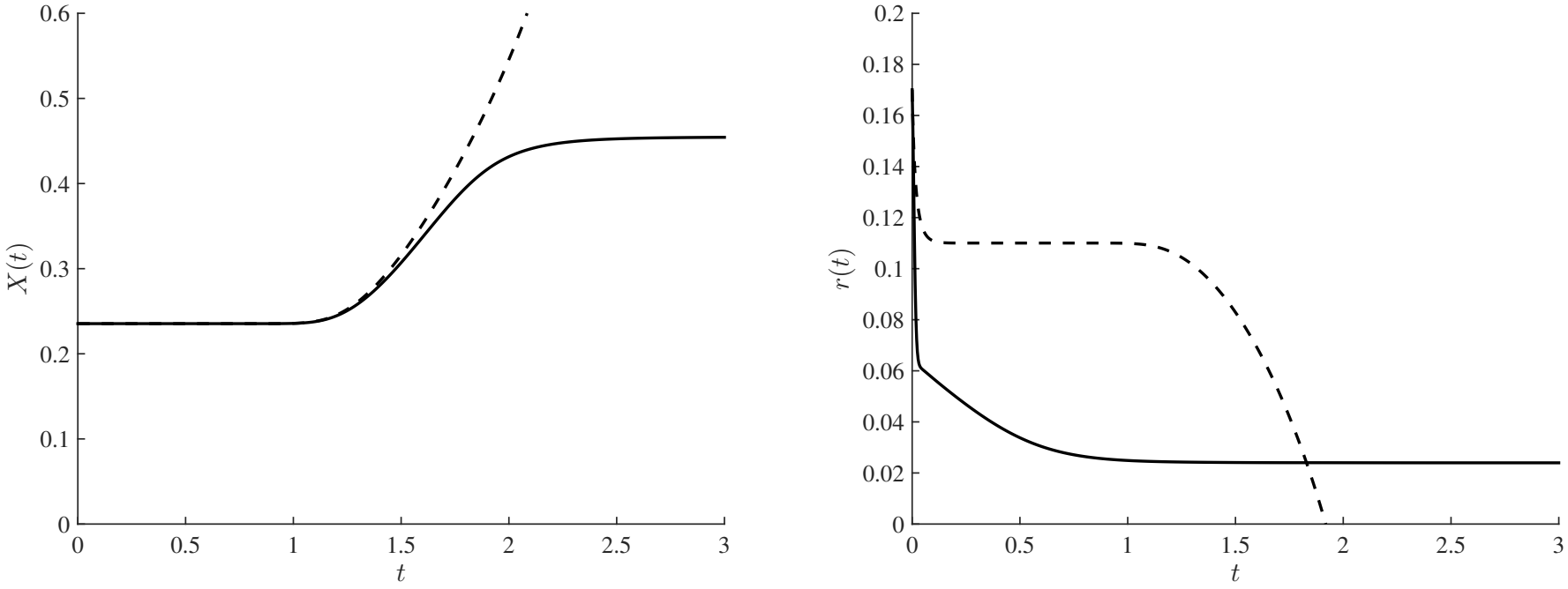

Figure: Uncompensated (dashed) and predictor-feedback law (solid) with P nominal law.

Quasilinear Transport PDE-ODE Interconnection

(w/ Miroslav Krstic)

Quasilinear Transport PDE-ODE Interconnection

$$
\begin{aligned}
\dot{X}(t) &= f\left(X(t), u(0,t)\right) \\
u_t(x,t) &= v\left(u(x,t)\right) u_x(x,t) \\
u(1,t) &= U(t).
\end{aligned}
$$

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$$

Assumptions:

$$
\begin{aligned}
\dot{X} &= f\left(X, \kappa\left(X\right) + \omega\right), \quad \text{ISS} \\
\dot{X} &= f\left(X, U\right), \quad \text{forward complete} \\
v &\in C^2\left(\mathbb{R}; \mathbb{R}_+\right)
\end{aligned}
$$

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Challenge:

$p(1,t)$ = value of the state at the time when control applied at t reaches the plant

$$= X\left(t + \underbrace{\frac{1}{v\left(u(1,t)\right)}}_{\substack{\text{prediction} \\ \text{horizon}}}\right)$$

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PDE predictor

$$
\begin{aligned}
p\left(x,t\right) &= X(t) + \int_0^x f\left(p(y,t), u(y,t)\right) \Gamma\left(u(y,t), u_y(y,t), y\right) dy, \\
& \quad x \in [0,1] \\
\Gamma\left(u(x,t), u_x(x,t), x\right) &= \frac{1}{v\left(u(x,t)\right)} - \frac{x v'\left(u(x,t)\right) u_x(x,t)}{v\left(u(x,t)\right)^2}
\end{aligned}
$$

PDE Predictor Feedback

$$
\begin{aligned}
U(t) &= \kappa\left(p\left(1,t\right)\right) = \kappa\left(X\left(t+\frac{1}{v\left(u(1,t)\right)}\right)\right) \\
p\left(x,t\right) &= X(t) + \int_0^x f\left(p(y,t),u(y,t)\right)\Gamma\left(u(y,t),u_y(y,t),y\right)dy, \\
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\end{aligned}
$$

Constant transport speed

$$
\begin{aligned}
U(t) &= \kappa\left(p\left(1,t\right)\right) = \kappa\left(X\left(t+\frac{1}{v}\right)\right) \\
p\left(x,t\right) &= X(t) + \frac{1}{v}\int_0^x f\left(p(y,t),u(y,t)\right)dy, \quad x\in[0,1]
\end{aligned}
$$

Theorem

$\exists\ \delta > 0$ *and* $\beta \in \mathcal{KL}$ *s.t.* $\forall\ X_0 \in \mathbb{R}^n$ *and* $u_0 \in C^1\,[0,1]$ *satisfying compatibility and*

$$|X(0)| + \|u(0)\|_{C^1} < \delta,$$

then

$$|X(t)| + \|u(t)\|_{C^1} \quad \leq \quad \beta\left(|X(0)| + \|u(0)\|_{C^1}, t\right), \quad t \geq 0$$

Coordinated Traffic Flow Control

Motivation

Coordinate available boundary actuators/sensors (e.g., via RM/VSL/VACS)

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First-order and **accounts** of drivers' look-ahead ability.

Consider

$$\begin{aligned} \rho_t(x,t) + \left(\rho(x,t)V\left(\rho(x,t)\right) - \epsilon\rho_x(x,t)\right)_x &= 0 \\ \rho(0,t) &= -U_0(t) \\ \rho\left(1,t\right) &= -U_1(t), \end{aligned}$$

with Greenshield's FD

$$V\left(\rho\right) = a\left(b - \rho\right)$$

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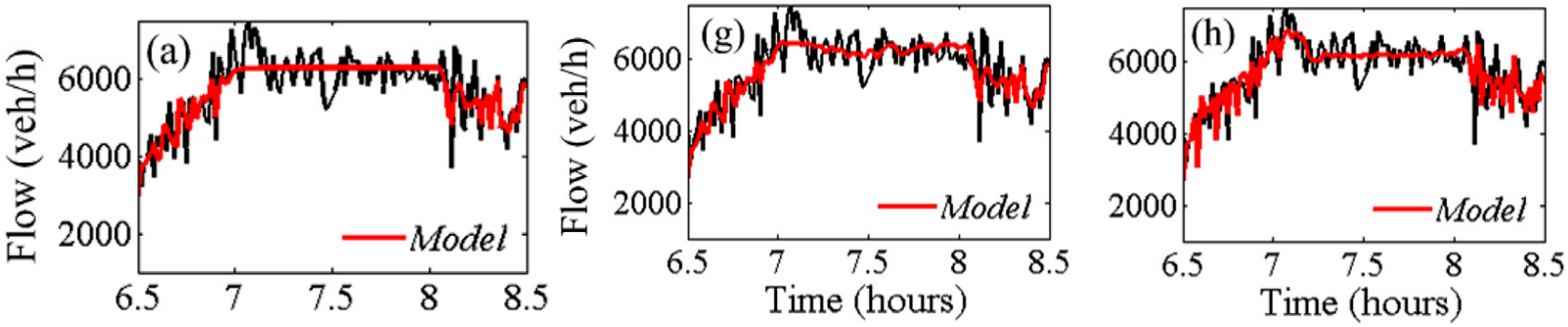

Figure: Left: LWR model. Middle: LWR with look ahead. Right: PW model.

PADECOT

PArtial Differential Equation model-based COntrol of Traffic flow

Kontorinaki et al., "First-order traffic flow models incorporating capacity drop: Overview and real-data validation," *TR-B*, 106: 52–75, 2017.

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with Greenshield's FD

$$V\left(\rho\right) = a\left(b-\rho\right)$$

Use alternative representation of **"label"** variable cumulative number of vehicles

$$u(x,t) = \int_x^1 \rho(y,t)dy + \int_0^t Q\left(\rho(1,s),\rho_x(1,s)\right)ds, \quad Q\left(\rho,\rho_x\right) = \rho V\left(\rho\right) - \epsilon\rho_x.$$

Viscous Hamilton-Jacobi PDEs

(w/ Rafael Vazquez)

Nonlinear Viscous Hamilton-Jacobi PDE

$$
\begin{aligned}
u_t(x,t) &= \epsilon u_{xx}(x,t) - a u_x(x,t)\,(b + u_x(x,t)) \\
u_x(0,t) &= U_0(t) \\
u_x(1,t) &= U_1(t),
\end{aligned}
$$

$\epsilon > 0$, $a, b \in \mathbb{R}$ (and $a \neq 0$), U_0, U_1 control inputs

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Solve the problems of **bilateral**

- Trajectory generation

  Find U_0^{r}, U_1^{r} generating $u^{\mathrm{r}}(x,t)$ s.t. $u^{\mathrm{r}}(x_0,t) = y_1^{\mathrm{r}}(t)$, $u_x^{\mathrm{r}}(x_0,t) = y_2^{\mathrm{r}}(t)$

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- Observer-based output-feedback trajectory tracking

  Find U_0, U_1 to achieve tracking of $u^{\mathrm{r}}(x,t)$, measuring $u(0,t)$, $u(1,t)$

Simulation

Set $a = b = 1$ (FD with $q_{\text{cap}} = \frac{1}{4}$, $\rho_{\text{cr}} = \frac{1}{2}$)

Regulate outlet density at ρ_{cr} and outlet flow $u_t(1,t)$ at q_{cap}

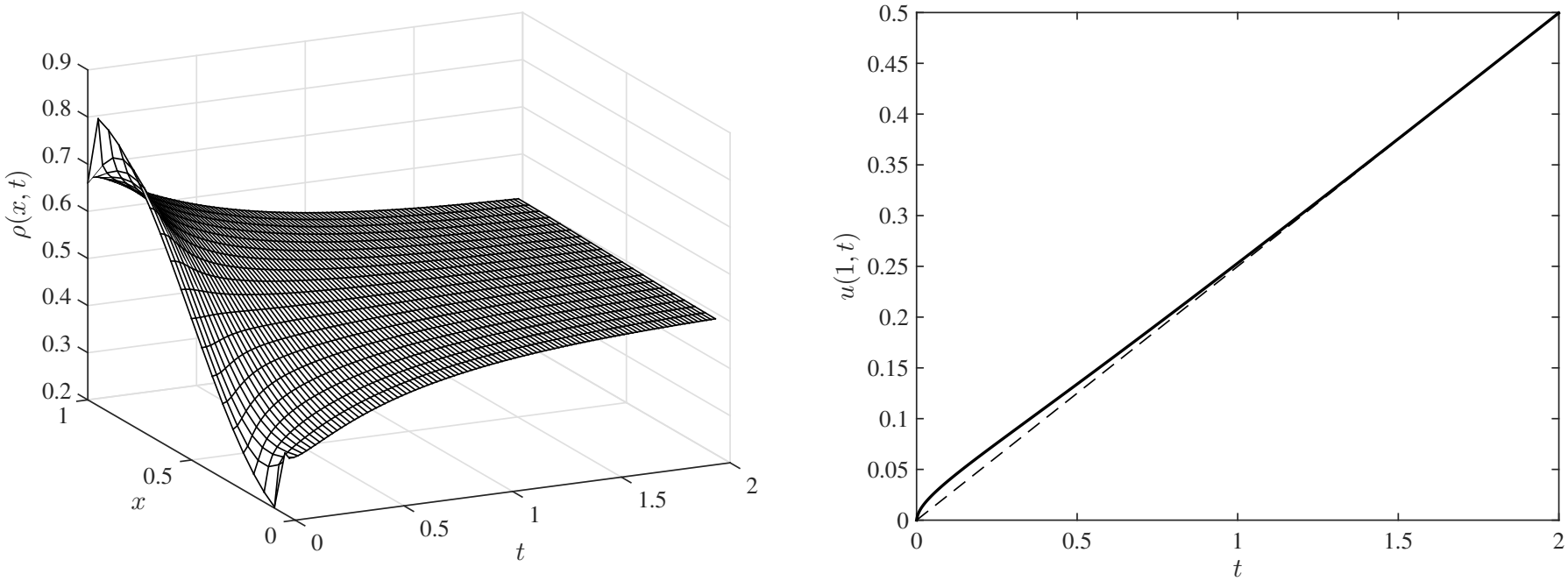

Figure: Left: Highway density. Right: Cumulative number of vehicles at the outlet.

Simulation

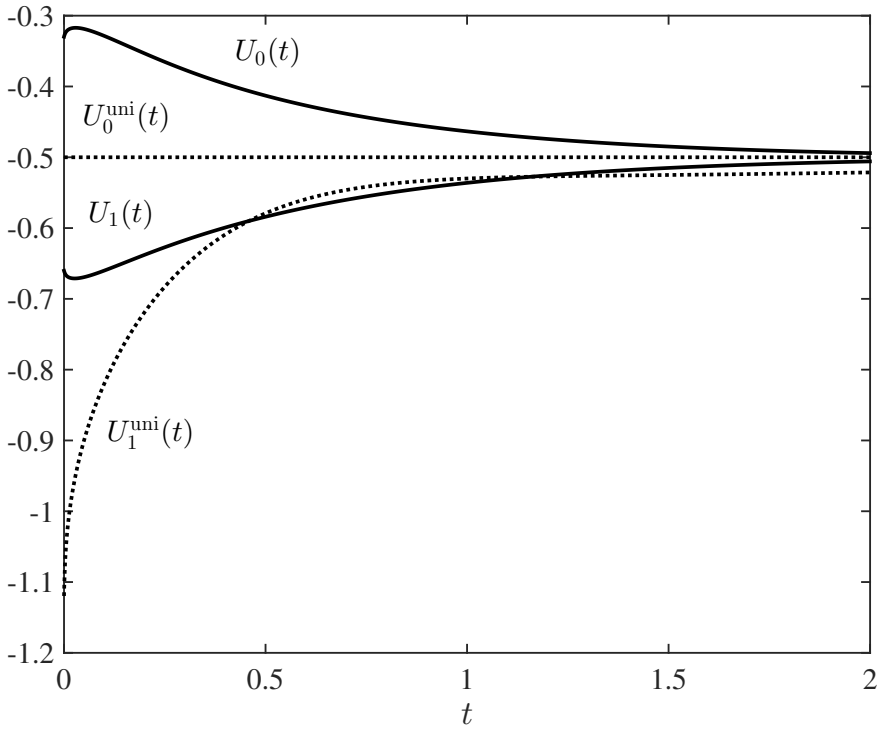

Figure: Bilateral (solid) and unilateral (dashed) controllers.

Trajectory Generation/Theorem

Let y_1^{r}, $y_2^{\mathrm{r}} \in G_{F,M,\gamma}$, *with* $1 \leq \gamma < 2$. $\exists\ \mu_1 > 0$ *s.t. if* $F \leq \mu_1$

$$
\begin{aligned}
u^{\mathrm{r}}(x,t) &= -\frac{\epsilon}{a}\ln\left(e^{\frac{ab}{2\epsilon}x}v^{\mathrm{r}}(x,t)+1\right) \\
U_0^{\mathrm{r}}(t) &= -\frac{\epsilon}{a}\frac{v_x^{\mathrm{r}}(0,t)+\frac{ab}{2\epsilon}v^{\mathrm{r}}(0,t)}{1+v^{\mathrm{r}}(0,t)}, \quad U_1^{\mathrm{r}}(t) = -\frac{\epsilon e^{\frac{ab}{2\epsilon}}}{a}\frac{v_x^{\mathrm{r}}(1,t)+\frac{ab}{2\epsilon}v^{\mathrm{r}}(1,t)}{1+e^{\frac{ab}{2\epsilon}}v^{\mathrm{r}}(1,t)},
\end{aligned}
$$

where

$$
\begin{aligned}
v^{\mathrm{r}}(x,t) &= \sum_{k=0}^{\infty}\frac{1}{\epsilon^k}\frac{(x-x_0)^{2k}}{(2k)!}\sum_{m=0}^{k}\binom{k}{m}\left(\frac{a^2b^2}{4\epsilon}\right)^{k-m}{y_{1,v}^{\mathrm{r}}}^{(m)}(t) \\
&\quad + \sum_{k=0}^{\infty}\frac{1}{\epsilon^k}\frac{(x-x_0)^{2k+1}}{(2k+1)!}\sum_{m=0}^{k}\binom{k}{m}\left(\frac{a^2b^2}{4\epsilon}\right)^{k-m}{y_{2,v}^{\mathrm{r}}}^{(m)}(t) \\
y_{1,v}^{\mathrm{r}}(t) &= e^{-\frac{ab}{2\epsilon}x_0}\left(e^{-\frac{a}{\epsilon}y_1^{\mathrm{r}}(t)}-1\right) \\
y_{2,v}^{\mathrm{r}}(t) &= e^{-\frac{ab}{2\epsilon}x_0}\left(-\frac{a}{\epsilon}e^{-\frac{a}{\epsilon}y_1^{\mathrm{r}}(t)}y_2^{\mathrm{r}}(t)-\frac{ab}{2\epsilon}\left(e^{-\frac{a}{\epsilon}y_1^{\mathrm{r}}(t)}-1\right)\right),
\end{aligned}
$$

satisfy the HJ PDE and $u^{\mathrm{r}}(x_0,t) = y_1^{\mathrm{r}}(t)$, $u_x^{\mathrm{r}}(x_0,t) = y_2^{\mathrm{r}}(t)$.

Observer Design

Copy of **linearized** error plant + **nonlinear** output injection

$$
\begin{aligned}
\hat{\tilde{v}}_t(x,t) &= \epsilon \hat{\tilde{v}}_{xx}(x,t) - \frac{a^2 b^2}{4\epsilon} \hat{\tilde{v}}(x,t) + p_2(x) \\
&\quad \times \left(\left(e^{-\frac{a}{\epsilon}\tilde{u}(0,t)} - 1 \right) e^{-\frac{a}{\epsilon} u^{\mathrm{r}}(0,t)} - \hat{\tilde{v}}(0,t) \right) \\
&\quad + p_1(x) \left(\left(e^{-\frac{a}{\epsilon}\tilde{u}(1,t)} - 1 \right) e^{-\frac{ab}{2\epsilon} - \frac{a}{\epsilon} u^{\mathrm{r}}(1,t)} - \hat{\tilde{v}}(1,t) \right) \\
\hat{\tilde{v}}_x(0,t) &= \tilde{V}_0(t) + p_{00} \left(\left(e^{-\frac{a}{\epsilon}\tilde{u}(0,t)} - 1 \right) e^{-\frac{a}{\epsilon} u^{\mathrm{r}}(0,t)} - \hat{\tilde{v}}(0,t) \right) \\
\hat{\tilde{v}}_x(1,t) &= \tilde{V}_1(t) + p_{11} \left(\left(e^{-\frac{a}{\epsilon}\tilde{u}(1,t)} - 1 \right) e^{-\frac{ab}{2\epsilon} - \frac{a}{\epsilon} u^{\mathrm{r}}(1,t)} - \hat{\tilde{v}}(1,t) \right)
\end{aligned}
$$

where

$$
\begin{aligned}
p_2(x) &= -\epsilon P_\xi(x,0), \quad p_1(x) = -\epsilon P_\xi(x,1) \\
p_{00} &= -P(0,0), \quad p_{11} = -P(1,1) \\
P(x,\xi) &= -\frac{1}{2}\sqrt{\frac{c_2}{\epsilon}} \frac{\mathrm{I}_1\left(\sqrt{\frac{c_2}{\epsilon}\left(\left(\xi - \frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2 \right)} \right)}{\sqrt{\left(\xi - \frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}} (\xi + x - 1)
\end{aligned}
$$

Observer-Based Output-Feedback Laws

$$
\begin{aligned}
U_0(t) &= -\frac{\epsilon}{a} e^{\frac{a}{\epsilon}\tilde{u}(0,t)} \left(\left(k(0,0) + \frac{ab}{2\epsilon} \right) \left(e^{-\frac{a}{\epsilon}\tilde{u}(0,t)} - 1 \right) \right. \\
&\quad \left. - e^{\frac{a}{\epsilon}u^{\mathrm{r}}(0,t)} \int_0^1 k_x(0,\xi)\, \hat{\tilde{v}}(\xi,t) d\xi \right) + U_0^{\mathrm{r}}(t) e^{\frac{a}{\epsilon}\tilde{u}(0,t)} \\
U_1(t) &= -\frac{\epsilon}{a} e^{\frac{a}{\epsilon}\tilde{u}(1,t)} \left(\left(k(1,1) + \frac{ab}{2\epsilon} \right) \left(e^{-\frac{a}{\epsilon}\tilde{u}(1,t)} - 1 \right) \right. \\
&\quad \left. + e^{\frac{ab}{2\epsilon} + \frac{a}{\epsilon}u^{\mathrm{r}}(1,t)} \int_0^1 k_x(1,\xi)\, \hat{\tilde{v}}(\xi,t) d\xi \right) + U_1^{\mathrm{r}}(t) e^{\frac{a}{\epsilon}\tilde{u}(1,t)},
\end{aligned}
$$

where

$$
k(x,\xi) = -\frac{1}{2}\sqrt{\frac{c_1}{\epsilon}} \frac{\mathrm{I}_1\left(\sqrt{\frac{c_1}{\epsilon}\left(\left(x - \frac{1}{2}\right)^2 - \left(\xi - \frac{1}{2}\right)^2 \right)} \right)}{\sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\xi - \frac{1}{2}\right)^2}} (x + \xi - 1)
$$

Theorem

Under conditions for y_1^{r}, y_2^{r}, $\exists\ \mu^* > 0$ and $\beta^* \in \mathcal{KL}$ *s.t. if* $\left(u_0, \hat{\tilde{v}}_0\right) \in H^2(0,1) \times H^2(0,1)$ *satisfy compatibility and*

$$\|\tilde{u}(t_0)\|_{H^1} + \|\hat{\tilde{v}}(t_0)\|_{H^1} \quad < \quad \mu^*,$$

then

$$\|\tilde{u}(t)\|_{H^1} + \|\hat{\tilde{v}}(t)\|_{H^1} \quad \leq \quad \beta^*\left(\|\tilde{u}(t_0)\|_{H^1} + \|\hat{\tilde{v}}(t_0)\|_{H^1}, t - t_0\right), \quad \textit{for all } t \geq t_0$$

Current & Future Steps

Extensions to CACC, nonlinear, and nonconstant-delay predictor-based designs

Current & Future Steps

Extensions to CACC, nonlinear, and nonconstant-delay predictor-based designs

2×2 quasilinear hyperbolic PDE/nonlinear ODE interconnection

Current & Future Steps

Extensions to CACC, nonlinear, and nonconstant-delay predictor-based designs

2×2 quasilinear hyperbolic PDE/nonlinear ODE cascade

HJ PDE/ODE cascades or HJ PDEs with actuator/sensor PDE/ODE dynamics

Thank you for attending