Repeat-Accumulate Codes

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Outline

Repeat-Accumulate Codes

- Divsalar et al. [1] attempted to prove AWGN coding theorems for a class of codes they call "turbo-like".
- Their proof technique used the ensemble input-output weight enumerator (IOWE) and combined this with the classical union bound to show that the ML word error probability reaches zero as $N \to \infty$ for some SNR threshold.
- The difficulty in calculating the IOWE restricted them to very simple coding systems which they called *repeat and accumulate* codes.

Repeat-Accumulate Codes

- The class of RA codes can be viewed as a subclass of LDPC codes (or Turbo codes)
- Their encoding is done as follows:
  1. A frame of information symbols of length N is repeated q times, resulting in a length qN frame.
  2. A random (but fixed) permutation is applied to the resulting frame.
  3. The permuted frame is fed to a rate-1 accumulator with transfer function $1/(1+D)$.

Repeat-Accumulate Codes

- Divsalar et al. prefer to think of the accumulator as a rate-1 block code whose input block $[x_1, \ldots, x_{qN}]$ and output block $[y_1, \ldots, y_{qN}]$ are related by the following formula:

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_1 + x_2 \\ &\cdots \\ y_n &= x_1 + x_2 + \ldots + x_{qN} \end{aligned}$$

- Which corresponds to the following generator matrix:

$$\mathbf{G_2} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}_{qN \times qN}$$

Repeat-Accumulate Codes

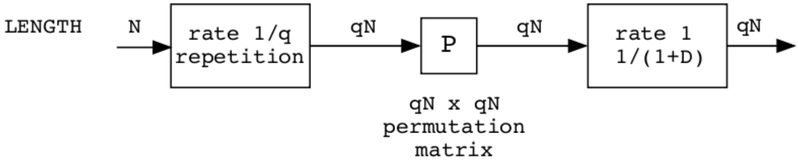


Figure: Encoder for an RA code

- The final stage of the encoder can be thought of either as an accumulator as pictured above (resulting in a "Turbo-like" code), or as the block code defined previously (resulting in an "LDPC-like" code).

Repeat-Accumulate Codes

- The resulting systematic generator matrix is:

$$\mathbf{G} = [\,\mathbf{I} \quad \mathbf{G}_1\mathbf{G}_2\,]$$

  where $\mathbf{G}_1$ is a $N \times qN$ matrix representing both the q-times repetition and the permutation of the N information bits.
- This encoding scheme results in a code of rate $1/(q+1)$.
- For a non systematic generator matrix we simply ommit the systematic part (i.e. the identity matrix):

$$\mathbf{G} = [\,\mathbf{G}_1\mathbf{G}_2\,]$$

- This encoding scheme results in a code of rate $1/q$.

Repeat-Accumulate Codes

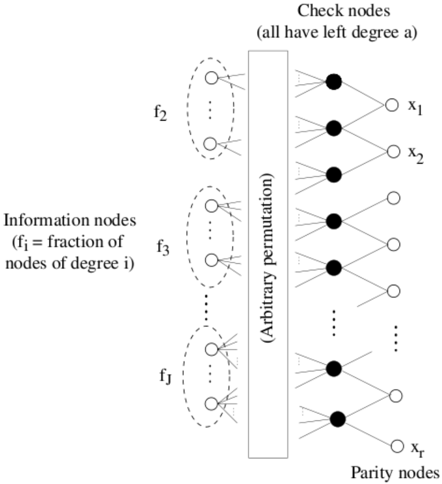
+ Simple structure.
+ Linear encoding complexity.
+ Efficient iterative decoding using belief propagation.
+ Good performance.

- High error floors.
- Small choice of rates and only rates below $1/2$ or equal to 1 (since $q \geq 1$).

Irregular Repeat-Accumulate Codes

- As with the superset of LDPC codes, irregular RA codes were introduced Jin et al. in 2000.
- Motivation: irregular LDPC codes generally perform a lot better than regular LDPC codes.
- Each information bit is not repeated a fixed number of times as with regular RA codes.

- The tanner graph of an IRA code with parameters $(f_1, \ldots, f_J, \alpha)$ where $f_i \geq 0$ and $\sum_i f_i = 1$ is as follows:



Irregular Repeat-Accumulate Codes

- The k variable nodes on the left are the information nodes.
- There are $r = (k \sum_i i f_i)/a$ check nodes.
- There are r variable nodes, called parity nodes, connected to the r check nodes in a simple zigzag manner.
- The recursive formula for the calculation of the parity bits is as follows:

$$x_j = x_{j-1} + \sum_{i=1}^{\alpha} v_{(j-1)\alpha+i}$$

Irregular Repeat-Accumulate Codes

- For the non systematic version of the above code, the codeword is:

$$(x_1, \ldots, x_r)$$

and the corresponding rate is:

$$\text{Rate} = \frac{k}{r} = \frac{\alpha}{\sum_i i f_i}$$

- For the systematic version of the above code, the codeword is:

$$(u_1, \ldots, u_k; x_1, \ldots, x_r)$$

and the corresponding rate is:

$$\text{Rate} = \frac{k}{r+k} = \frac{\alpha}{\alpha + \sum_i i f_i}$$

Irregular Repeat-Accumulate Codes

- As with LDPC codes, density evolution can be used to find good degree distributions by solving a linear program.
- By using the Gaussian approximation for the messages exchanged by the BP algorithm, the problem is greatly simplified.
- The best rate-1/2 IRA code by Divsalar et al. [2] has a threshold of 0.266 dB, while the best rate-1/2 irregular LDPC code found in [3] has a threshold of 0.25 dB (both calculated by using exact density evolution).

Irregular Repeat-Accumulate Codes

- + Simple structure.
- + Linear encoding complexity.
- + Efficient iterative decoding using belief propagation.
- + Good performance.
- + Disadvantages of regular RA codes are fixed.

Structured IRA Codes

- The parity-check matrix of an RA code can be written as follows:

$$\mathbf{H} = [\, \mathbf{H}_1 \quad \mathbf{H}_2 \,]$$

where

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 \end{bmatrix}$$

Structured IRA Codes

- We define the matrix $\mathbf{P}$ as follows:

$$\mathbf{P} = \begin{bmatrix} \pi^{b_{0,0}} & \pi^{b_{0,1}} & \dots & \pi^{b_{0,J-1}} \\ \pi^{b_{1,0}} & \pi^{b_{1,1}} & \dots & \pi^{b_{1,J-1}} \\ \vdots & \vdots & \vdots & \vdots \\ \pi^{b_{L-1,0}} & \pi^{b_{L-1,1}} & \dots & \pi^{b_{L-1,J-1}} \end{bmatrix}_{L \times J}$$

  where π is a right cyclic shift $Q \times Q$ permutation matrix and $b_{i,j} \in \{0, 1, \dots, Q-1, \infty\}$ are the corresponding exponents.
- We define π^{∞} as the zero matrix.

Structured IRA Codes

- If we use $\mathbf{H}_1 = \mathbf{P}$, the resulting code has a poor minimum distance.
- We use a permuted version of $\mathbf{P}$ instead, where the permutation is chosen so that the minimum codeword weight for low-weight inputs is increased.
- So, the final parity-check matrix will be:

$$\mathbf{H} = [\, \mathbf{\Pi P} \quad \mathbf{H}_2 \,]$$

- The resulting code is regular unless we choose to mask out some entries (by choosing $b_{i,j} = \infty$) in the $\mathbf{P}$ matrix in accordance with a targeted repetition profile.

Accumulate-Repeat-Accumulate Codes

- In [5], the addition of a rate-1 accumulator before the repetition code was proposed.
- Using density evolution, it was shown that this precoding improves performance.
- For example, for a rate-$1/3$ RA code, the iterative decoding threshold is 0.73 dB, while with the addition of the precoder the threshold is lowered to -0.048 dB.
- This improvement in performance is called *precoding gain*.

Accumulate-Repeat-Accumulate Codes

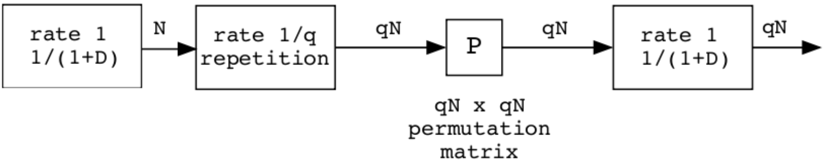


Figure: Encoder for an ARA code.

- Irregular ARA codes (IARA) were also proposed.
- Rates greater than $1/2$ can be achieved by puncturing.

References

[1] D. Divsalar, H. Jin, and R. J. McEliece, "Coding theorems for 'turbo-like' codes", pp. 201-210 in Proc. 36th Allerton Conf. on Communications, Control, and Computing (Sept. 1998).

[2] H. Jin, A. Khandekar, and R. McEliece, "Irregular repeat-accumulate codes", in Proc. 2nd International Symposium on Turbo Codes, pp. 1-8, 2000.

[3] T. J. Richardson, A. Shokrollahi, and R. Urbanke, "Design of provably good low-density parity-check codes", submitted to IEEE Trans. Inform. Theory.

[4] Y. Zhang, W. E. Ryan, "Structured IRA Codes: Performance Analysis and Construction", IEEE Trans. Comm., Vol. 55, No. 5, pp. 837-844, May 2007

[5] A. Abbasfar, D. Divsalar, and K. Yao, "Accumulate-Repeat-Accumulate Codes", IEEE Trans. Comm., Vol. 55, No. 4, April 2007