Trellises for Linear Block Codes

DESPOINA GEORGIADOU

Chania 2008

Outline

The FSM Model

For every block code we know that:

- it has finite memory, that stores input information for a certain finite time interval
- symbols stored in memory and the current input affect the output code symbols according to a certain encoding rule
- at any encoding time symbols stored in memory specify a state of the encoder at that time instant
- since encoder's memory has finite size, the allowable states are also finite
- when new symbols are shifted in memory, some old are shifted out of it causing a **state transition**

So, an encoder can be modeled as a finite-state machine (FSM).

This dynamic behaviour of a encoder can be graphically represented by a trellis diagram !

The FSM Model

Let a_i be the encoder's input bit at time-i, u_j the output bit at time-i and g_j the rows of the generator matrix, then:

$$(a_0, a_1, ..., a_{k-1})^* \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{k-1} \end{pmatrix} = (a_0 g_0 + a_1 g_1 + ... + a_{k-1} g_{k-1})$$

$$= \begin{array}{llll} [a_0 g_{00} & a_0 g_{01} & \cdots & a_0 g_{0(n-1)}] \\ [a_1 g_{10} & a_1 g_{11} & \cdots & a_1 g_{1(n-1)}] \\ \vdots & & & \\ +[a_{k-1} g_{(k-1)0} & a_{k-1} g_{(k-1)1} & \cdots & a_{k-1} g_{(k-1)(n-1)}] \end{array}$$

$$= [\sum_{i=0}^{n-1} a_{k-1} g_{(k-1)0} \cdots \sum_{i=0}^{n-1} a_{k-1} g_{(k-1)(n-1)}]$$

$$= (u_0, u_1, ..., u_{n-1})$$

About trellis

Last time we talked about states (or nodes or vertices), branches and their labels.

Today we define:

Γ = {0,1,2,...,n}: the entire encoding interval (span), is a sequence of all encoding time instants

state space $\sum_i(C)$: all allowable states at a given time instant i

state space dimension at time-i: $p_i(C) = log_2|\sum_i(C)|$

state space complexity profile: the sequence $\{|\sum_0(C)|, |\sum_1(C)|, ..., |\sum_n(C)|\}$

state space dimension profile: the sequence $(p_0, p_1, .., p_n)$

Trellis Oriented form of G

A generator matrix G is in Trellis Oriented Form (TOF) when for each row g $\epsilon \{g_0, g_1, ..., g_{k-1}\}$

- The leading 1[a] appears in a column before the leading 1 of any row below it.
- No two rows have their trailing 1[b] in the same column.

[a] the first nonzero component of a row
[b] the last nonzero component of a row

For G_{TOGM} (trellis oriented generator matrix) we have:

1. digit (or bit) span of g, ϕ(g) = {i,..,j}, is the smallest index interval containing all nonzero components of g.
2. time span of g, τ(g)= {i,..,j+1}, same as bit span in terms of time
3. Active time span, $\tau_a(g)$= [i+1,j], for j>i.

Examples

Every generator matrix can be put in TOF (by elementary row operations) which is not necessarily systematic form.

Matrix $G = \begin{pmatrix} 1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1 \\ 0&0&1&1&0&0&1&1 \\ 0&1&0&1&0&1&0&1 \end{pmatrix}$ is not in TOF , whereas

$G_{TOGM} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 1&1&1&\mathbf{1}&0&0&0&0 \\ 0&1&0&1&1&0&\mathbf{1}&0 \\ 0&0&1&1&0&\mathbf{1}&0&0 \\ 0&0&0&0&1&1&1&\mathbf{1} \end{pmatrix}$ is. The G_{TOGM} matrix is the G with the second and the fourth rows interchanged and the fourth row added to all the others.

Bit span an active time span example

- $\phi(g_3) = [4,7]$
- $\tau_a(g_3) = [5,7]$
- $\tau(g_3) = [4,8]$

Partitioning matrix G_{TOGM} of a (n,k) linear code C

At time-i , $0 \leq i \leq n$ we partition the rows of G_{TOGM} into:

1. G_i^p : rows with bit span in the interval [0,i-1]
2. G_i^f : rows with bit span in the interval [i,n-1]
3. G_i^s : rows whose active time spans contain time-i

We can respectively partition information bits a_0,a_1,...,a_{k-1} into:

1. A_i^p : bits that don't affect the encoder output after time-i
2. A_i^f : bits that affect the encoder output only after time-i
3. A_i^s : bits that affect the output both before and after time-i

So A_i^s is the encoder's memory (or state) at time-i and $|A_i^s| = p_i$!

Partitioning matrix G_{TOGM} of a (n,k) linear code C - Example

So, for $G_{TOGM} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 1&1&1&1&0&0&0&0 \\ 0&1&0&1&1&0&1&0 \\ 0&0&1&1&0&1&0&0 \\ 0&0&0&0&1&1&1&1 \end{pmatrix}$ we have:

Example

Time	G_i^p	G_i^f	G_i^s	p_i
0	ϕ	g_0,g_1,g_2,g_3	ϕ	0
1	ϕ	g_1,g_2,g_3	g_0	1
2	ϕ	g_2,g_3	g_0,g_1	2
3	ϕ	g_3	g_0,g_1,g_2	3
4	g_0	g_3	g_1,g_2	2
5	g_0	ϕ	g_1,g_2,g_3	3
6	g_0,g_2	ϕ	g_1,g_3	2
7	g_0,g_1,g_2	ϕ	g_3	1
8	g_0,g_1,g_2,g_3	ϕ	ϕ	0

We also mention that:

g^* : the row in G_i^f whose leading 1 is at bit position i

- its uniqueness is guaranteed, but the existence is not.

a^* : information bit that corresponds to row g^* (current input information bit)

The output code bit generated between time-i and time-(i+1) is:

$u_i = a^* + \sum_{l=1}^{p_i}(a_l^{(i)} * g_{l,i}^{(i)})$ [a], if g^* exists

$u_i = \sum_{l=1}^{p_i}(a_l^{(i)} * g_{l,i}^{(i)})$, if g^* doesn't exist [b]

---

[a] $g_{l,i}^{(i)}$ is is the ith component of $g_l^{(i)}$ in G_i^s

[b] in this case we can put $a^* = 0$ (dummy information bit)

Output code bit - example

For G_{TOGM} = $\begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix}$ = $\begin{pmatrix} 1&1&1&1&0&0&0&0 \\ 0&1&0&1&1&0&1&0 \\ 0&0&1&1&0&1&0&0 \\ 0&0&0&0&1&1&1&1 \end{pmatrix}$ we have:

- $G_2^s = \{g_0, g_1\}$ and $A_2^s = \{a_0, a_1\}$ (a_0, a_2 define the encoder's state at time-2)
- $g^* = g_2$ and $a^* = a_2$ (a_2 is the current input)
- $u_2 = a^* + \sum_{l=1}^{p_2}(a_l^{(2)} * g_{l,2}^{(2)}) = a_2 + a_0 g_{02} + a_1 g_{12} = a_2 + a_0 * 1 + a_1 * 0 = a_2 + a_0$

We also mention that:

g^0 : the row in G_i^s whose trailing 1 is at bit position i

- g_i^0 is the last nonzero component of g^0 (if it exists)

a^0 : information bit in A_i^s that corresponds to row g^0 (current input information bit)

- it is the oldest bit information in memory at time-i

At time-(i+1) we have:

$G_{i+1}^s = (G_i^s \backslash \{g^0\}) \cup g^*$, if g^0 exists
$A_{i+1}^s = (A_i^s \backslash \{a^0\}) \cup a^*$, if a^0 exists

Construction of a bit-level trellis

State labeling

In a code trellis, each state is labeled based on the information set that defines the state space at a particular encoding time instant.

The label l(s) of a state s is set to zero except for the components at the positions corresponding to the information bits in $A_i^s = \{a_1^i, a_2^i, ..., a_{p_i}^i\}$.

Thus, the label of the state s_i is: $l(s_i) = (a_1^i, a_2^i, ..., a_{p_i}^i)$.

Labeling Example

If at time i=4 we have $A_4^s = \{a_1, a_2\}$, then l(s_4)=(0,a_1, a_2,0).

Construction of a bit-level trellis

Now we are ready to construct a trellis diagram:

To construct the bit-level trellis diagram, all we need is:

- G_{i+1}^{s}
- A_{i}^{s}
- A_{i+1}^{s}

Construction of a bit-level trellis with generator matrix

The construction steps are:

1. Determine G^s_{i+1} and A^s_{i+1}.
2. Form the state space $\sum_{i+1}(C)$ at time-(i+1).
3. For each state $s_i \in \sum_i(C)$, determine its transition(s) based on the change from A^s_i to A^s_{i+1} (bits a^0 and a^*).
4. Connect s_i to its adjucent state(s) in $\sum_{i+1}(C)$ by branches.
5. For each transition, determine the output code bit u_i.
6. Use this u_i to label the corresponding branch.

Construction of a bit-level trellis

So for $G_{TOGM} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 1&1&1&1&0&0&0&0 \\ 0&1&0&1&1&0&1&0 \\ 0&0&1&1&0&1&0&0 \\ 0&0&0&0&1&1&1&1 \end{pmatrix}$ we have:

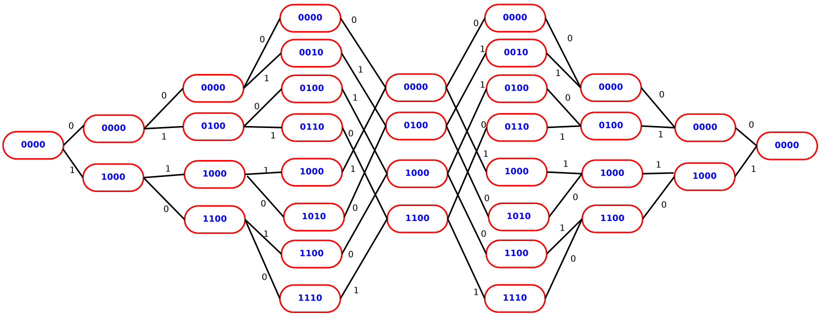
Example

Time	G_i^s	a^*	a^0	A_i^s	State Label
0	ϕ	a_0	-	ϕ	(0000)
1	g_0	a_1	-	a_0	($a_0$000)
2	g_0,g_1	a_2	-	a_0,a_1	($a_0 a_1$00)
3	g_0,g_1,g_2	-	a_0	a_0,a_1,a_2	($a_0 a_1 a_2$0)
4	g_1,g_2	a_3	-	a_1,a_2	(0$a_1 a_2$0)
5	g_1,g_2,g_3	-	a_2	a_1,a_2,a_3	(0$a_1 a_3 a_3$)
6	g_1,g_3	-	a_1	a_1,a_3	(0$a_1$0a_3)
7	g_3	-	a_3	a_3	(000a_3)
8	ϕ	-	-	ϕ	(0000)

Construction of a bit-level trellis with generator matrix

Computing all u_i for labeling the branches, we come to the end!

So, the trellis we obtain is:



Trellis Complexity

A trellis' complexity is:

the branch complexity: the number of branches

the state complexity: the maximum state space dimension $p_{max}(C)$

A minimal trellis

The trellis T is said to be minimal if for any other n-section trellis T' for C with state space dimension profile $(p'_0, p'_1, p'_2, ..., p'_n)$ the following inequality holds: $p_i \leq p'_i$, for $0 \leq i \leq n$.
The minimal trellis has the minimal total number of states and is also a minimal branch trellis (with smallest branch complexity).

Parallel Decomposition For Minimal Trellis

Why?

A minimal code trellis is generally densely connected. So there is a difficulty in implementation.

To address that problem, we can decompose the trellis into parallel and structurally identical subtrellises of smaller dimensions without cross-connections between them.

Decomposition of minimal trellis into minimal parallel subtrellises

We define the following index set:
$I_{max(C)}$={i:$p_i(C) = p_{max}(C)$, for 0$\leqslant i \leqslant n\}$

Theorem

If there exists a row g in the G_{TOGM} for an (n,k) linear code C such that $\tau_a(g) \supseteq I_{max}(C)$, then the subcode C_1 of C generated by $G_{TOGM} \diagdown \{g\}$ has a minimal trellis T_1 with the maximum state space dimension $p_{max}(C_1) = p_{max}(C) - 1$, and $I_{max(C_1)} = I_{max(C)} \cup \{i : p_i(C) = p_i(max) - 1, i \text{ not in } \tau_a(g)\}$.

If G is in TOF, $G_1 = G \backslash \{g\}$ is also in TOF. If theorem holds, then we can decompose C into two parallel and structurally identical subtrellises, one for C_1 and the other for the coset $C_1 \oplus g$.

Parallel Decomposition-Example

example

For $G_{TOGM} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 1&1&1&1&0&0&0&0 \\ 0&1&0&1&1&0&1&0 \\ 0&0&1&1&0&1&0&0 \\ 0&0&0&0&1&1&1&1 \end{pmatrix}$ we have: $I_{max}(C)$=[3,5] as state space dimension is (0,1,2,3,2,3,2,1,0) and $p_{max}(C)$=3. The only row whose active time span τ_a contains $I_{max}(C)$ is g_1 where $\tau_a(g_1)$=[2,6].

So, $G_1 = \begin{pmatrix} 1&1&1&1&0&0&0&0 \\ 0&0&1&1&0&1&0&0 \\ 0&0&0&0&1&1&1&1 \end{pmatrix}$

Parallel Decomposition-Example

Constructing the trellis for both C_1 and $g_1 \oplus C_1$ we get the following parallel decomposition: