Introduction to Game Theory

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Game Theory

- Game theory is the analysis of conflict and cooperation among intelligent rational decision makers.
- A decision maker is said to be rational if he makes decisions consistently in a pursuit of his own objectives.
- In a game, two or more individuals make decisions that influence each others expected utility.
- The decision makers are called the players.
- The decision objects of players are generally called strategies.

Game Theory

- Game theory offers two basic tools
  - Models of games.
  - Solution concepts.
- There is a variety of models that represent different scenarios that might show up in real-life situations.
- Solution concepts are predictions about what rational intelligent players should play.

Game Theory

- Game theory has its origins in social sciences. The rational intelligent assumption is not always true when we refer to human beings.
- We are interested in games that represent conflict of interests among wireless nodes, such as software defined radios.
- SDRs are programmable devices that act according to their programming so they can be considered rational.
- The behavior of a wireless device may affect the communication capabilities of neighboring devices because the wireless medium is usually shared in wireless networks.

Game Theory

- Game theory can be seen as an extension of decision theory to the case of many decision makers.
- We will introduce the concept of utility functions which is of outmost importance in game theory and then proceed with examples of real game models.

Utility Theory

- Let X be a set of possible outcomes or alternatives that a decision making entity wants to select from.
- Some outcomes might be more preferable than others.
- A binary relation R on X is any subset of $X \times X$.
- If $(x, y) \in R$ we write xRy
- Let $\succeq$ be a binary relation on X, for which $x \succeq y$, if outcome x is at least as preferable as outcome y.
- $\succeq$ defines a preference relation if it is complete and transitive.

Utility Theory

- $\succeq$ is complete if for every $x, y \in X$, $x \succeq y$ or $y \succeq x$.
- Assume x is a 100kbps connection with 1ms delay and y is a 10Mbps connection with 100ms delay.
- The first would be good for transmitting realtime voice the other for transmitting stored video.
- The preference depends on the application.
- the preferences of the user in the application layer define preference relations in lower layers.

Utility Theory

- It would be convenient if we could represent $\succeq$ using numbers.
- We will represent the relation $\succeq$ with a function, $u : X \rightarrow \Re$, for which, $x \succeq y \Leftrightarrow u(x) \geq u(y)$.
- Utility functions are not unique. Any composition of a utility function with a strictly increasing function is a utility function that represents the same preference relation.
- For finite or even countable infinite X such a function always exists.
- For uncountable infinite X its not always possible to find a utility function.

Utility Theory

- We say $\succeq$ is continuous if for all $\{x_n\}$ such that $\{x_n\} \rightarrow x$
  1. $\forall n, x_n \succeq y \Rightarrow x \succeq y$
  2. $\forall n, y \succeq x_n \Rightarrow y \succeq x$
- The relation $\succeq$ is continuous iff there exists a continuous utility function $u : X \rightarrow \Re$ that represents it.

Utility Theory

- In realistic situations the outcome of a game may depend not only on the decision maker's actions but on random events too.
- Let Z denote the set of all possible outcomes.
- Let $\Delta(Z)$ denote the set of probability distributions over the set Z.
- a lottery is any member of $\Delta(Z)$.
- A decision maker must express preferences over lotteries.

Utility Theory

- We could represent a preference over lotteries with a utility function just like before.
- But we are mostly interested in the so called expected utility representations.
- if $\succeq$ satisfies certain axioms, then it can be proven that there exists a utility function $u : Z \rightarrow \Re$ such that $\forall p, q \in \Delta(Z), p \succeq q \Leftrightarrow E_p(u(z)) \geq E_q(u(z))$
- Thus, a rational decision maker should make decisions that maximize a certain expected utility.

Types of Games

- Games can be partitioned into categories based an various criteria
- If a game is played just once and players get their payoffs at the end, then the game is static
- If there are many rounds, at the end of each the players get a payoff, then the game is said to by dynamic
- If all players know all the utility functions, then the game is said to be with complete information
- If there is some information concerning the game that is not common knowledge, then the game is said to be with incomplete information

Types of Games

- If players aim to maximize their own utility then the game is said noncooperative.
- If players are allowed to form coallitions the game is said to be cooperative.
- We will focus on non-cooperative game theory in this presentation.

Static Games of Complete Information

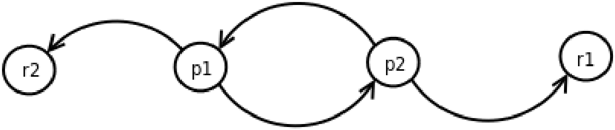
- In normal or strategic form a static game of complete information is represented by a triple $\langle N, \{S_i\}_{i \in N}, \{u_i(.)\}_{i \in N} \rangle$
- $N = \{1, \ldots, n\}$ is the set of players.
- S_i is the set of strategies of player i.
- $u_i :\longrightarrow \Re$ is player's i utility function.

Static Games of Complete Information

- A combination of players' strategies form a strategic profile $s = (s_1, \ldots, s_n)$
- The set of all strategic profiles is $S = S_1 \times S_2 \times \cdots \times S_n$
- we represent the set of strategies of all players except i with s_{-i}.
- We assume all players select their strategies $s_i \in S_i$ simultaneously.
- Each player wants to maximize his own utility.

Static Games of Complete Information

Example: Forwarder's Dilemma



- Player 1 wants to send to r1 using p2 as forwarder.
- Player 2 wants to send to r2 using p1 as forwarder.
- Forwarding has a cost c.
- If player 1 forwards player's 2 packet, player 2 gets a reward 1 and vice versa.
- Each player's utility is his reward minus the cost.

Static Games of Complete Information

The Forwarder's Dilemma game in strategic

P1 /P2	F	D
F	1-c,1-c	-c,1
D	1,-c	0,0

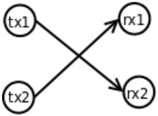
- Strategy F is to forward other player's packet.
- Strategy D is to drop other player's packet.

Static Games of Complete Information

- A strategy s_i for player i is said to be **strictly dominated** if there exists some other strategy $s_i' \in S_i$ such that $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \forall s_{-i} \in S_{-i}$.
- In forwarder's dilemma F is a strictly dominated strategy for both players.
- A rational player would never choose a strictly dominated strategy.
- For the forwarder's dilemma, (D,D) is the only possible outcome.

Static Games of Complete Information

Example: Random Access Game



- Two transmitters share the same medium.
- When a player transmits, he pays a cost c.
- If the other player remains silent, he gets a reward 1.
- If both players transmit simultaneously, there is 0 reward.

Static Games of Complete Information

The Random Access Game in strategic form

P1 /P2	Q	T
Q	0,0	0,1-c
T	1-c,0	-c,-c

- Strategy T is to transmit.
- Strategy Q is to remain silent.
- There are no strictly dominated strategies.

Static Games of Complete Information

- If we allow players to randomize over their strategies then we get the **mixed extension** of the game.
- A **mixed strategy** for player i is any pdf σ_i on the set S_i.
- If players play the mixed strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$, then the expected utilities they get are $u_i(\sigma) = \sum_{s \in S} u_i(s)\sigma(s)$
- usually, we assume that players choose strategies independently so $\sigma(s) = \prod_{i=1}^{n} \sigma_i(s_i)$

Static Games of Complete Information

- Example: in the Random Access Game, if player 1 chooses T with probability p and player 2 chooses T with probability q then utilities are:
  - $u_1(p,q) = q(1-p)(1-c) - (1-q)(1-p)c$
  - $u_2(p,q) = p(1-q)(1-c) - (1-p)(1-q)c$

Static Games of Complete Information

- A solution concept is a prediction of what strategic profiles might be actually played if players are rational and intelligent.
- We can use strict dominance to iteratively eliminate strategies from the game. Those strategies that survive iterative strict dominance are called the **strictly undominated strategies**.
- The set of undominated strategies however might be very large.
- The most widely used solution concept is the **Nash Equilibrium**
- A strategic profile s^* is called a NE if for $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \forall i \in N, \forall s_i \in S_i$

Static Games of Complete Information

- A NE is a stable outcome of a game meaning that if all players were to play the strategies in s^*, none would have an incentive to unilaterally deviate.
- However two or more players might have an incentive to deviate together.
- A NE in which no set of players has any incentive to deviate is called a strong NE.
- A game may have none, one or many NE

Static Games of Complete Information

P1 /P2	F	D
F	1-c,1-c	-c,1
D	1,-c	0,0

- The only NE of Forwarder's Dilemma is (D,D).
- Both players would be better off if they choose (F,F).
- A NE is not necessarily a global optimum.

Static Games of Complete Information

- An outcome of a game is **weakly Pareto efficient** iff there is no other outcome that would make all players better off.
- An outcome of a game is **strong Pareto efficient** iff there is no other outcome that would make at least one player better off without reducing the utilities of the rest.
- A NE is not always Pareto efficient.

Static Games of Complete Information

P1 /P2	Q	T
Q	0,0	0,1-c
T	1-c,0	-c,-c

- The random access game has two NE (T,Q) and (Q,T).
- Both are Pareto efficient.
- They are unfair though!

Static Games of Complete Information

- An equivelant way to define a NE is using the best-reply correspondence
- a point-to-set mapping $M_i(s)$ that associates each strategy profile s with a subset of S_i that maximize players i utility given the strategies in s_{-i} is said to be the best-reply correspondence for player i.
- The best-reply correspondence of the game is $M = \times_{i \in N} M_i(s)$
- A strategic profile s is a NE iff $s \in M(s)$.

Static Games of Complete Information

- For the forwarder's dilemma game $M_i(s) = D, \forall s \in S$.
- For the Random Access Game

$$M_i(s) = \begin{cases} T & \text{if } s_{-i} = Q \\ Q & \text{if } s_{-i} = T \end{cases}$$

Static Games of Complete Information

- A NE exists if the best-reply correspondence has a fixed point.
- There are theorems that gives sufficient conditions for the existance of fixed points of a correspondence and therefore for the existance of NE.
- One such theorem states that if every players actionn space is a compact convex set in Euclidean space and the utility functions are continuous in S and quasi-concave then the game has at least one pure NE.
- Nash proved in 1956 that every finite game has a NE in mixed strategies.

Static Games of Incomplete Information

- In a game of complete information everything is assumed to be common knowledge.
- A more realistic model might assume that each player has some private information.
- For instance, each node in a wireless network knows his own channel.
- In game theory literature the private information of a player is called its **type**.
- Games of incomplete information are represented by **Bayesian Games**.

Static Games of Incomplete Information

- A Bayesian Game consists of the following:
  - a set of players N
  - a set of types T_i for each player
  - set of actions C_i for each player
  - a probability function $p_i(\cdot, t_i)$ for each player
  - a utility function $u_i(c, t)$

Static Games of Incomplete Information

- Each player is assumed to know his own type.
- We use the term actions for the decision objects of players instead of strategies. A strategy for player i is a function from the players types T_i to his actions C_i.
- The probability function $p_i(\cdot, t_i)$ is a function from T_i into $\Delta(T_{-i})$.
- It represents what player i knows about other players types when his own type is t_i.

Static Games of Incomplete Information

- A Bayesian Game can be represented in strategic form as follows:
  - The set of players is $T^* = \bigcup_{i \in N} T_i$.
  - The strategies available for each of the players that represent player i of the Bayesian Game are $D_{t_i} = C_i$.
  - For any d in $\times_{s \in T^*} D_s$ the utility function is defined as $u_{t_i}(d) = \sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) u_i((d(t_j))_{j \in N}, (t_j)_{j \in N})$.

Dynamic Games of Complete Information

- When players interact by playing a similar stage game numerous times, the game is called a dynamic, or repeated game. Unlike static games, players have at least some information about the strategies chosen on others and thus may contingent their play on past moves.
- Each time they play they get a payoff.
- Players express their preferences over sequences of payoffs.

Dynamic Games of Complete Information

- A very general model of a repeated game is of the form $\Gamma^r = (N, \Theta, (D_i, S_i, u_i)_{i \in N}, q, p)$ where
- N is the set of players
- Θ is the set off the possible states of nature as it is described in game theory text books
- For each player i, the sets D_i and S_i, denote the set of moves player i can choose and the set of signals he may receive, at each round of the game
- q is an initial distribution in $\Delta(S \times \Theta)$
- p is a transition function $p : D \times \Theta \rightarrow \Delta(S \times \Theta)$
- $u_i : D \times \Theta \rightarrow \Re$ is the payoff function of player i

Dynamic Games of Complete Information

- The games is played for infinite rounds.
- In each round some state in Θ is the current state of the world.
- At the begining of each round, each player receives a signal s_i that represents his observation of the other players moves in the previous round.
- Each player uses all his past observations to choose his next move.
- A strategy is a plan of what to do in each round as a function of the history of the game.

Dynamic Games of Complete Information

- There are many ways in which players may define their preference among different payoff sequences.
- The simplest would be to assume that players aim at maximizing their sum of payoffs.
- For infinitely repeated games though this could be infinite.
- An alternative way that doesn't suffer from the previous problem is the δ-discounted average.
- If the sequence of payoffs player i gets are $(w_i(1), w_i(2), \dots)$ the δ-discounted average is

$$(1-\delta)\sum_{k=1}^{\infty}\delta^{k-1}w_i(k)$$

Dynamic Games of Complete Information

- Nash equilibria are defined for repeated games just like for static games.
- There is also a stronger equilibrium concept in repeated games called subgame-perfect equilibrium.
- A strategic profile is a **subgame-perfect equilibrium** if it is a NE of every subgame.
- In a subgame-perfect equilibrium there is no incentive for players to deviate from the defined moves at each round.

Dynamic Games of Complete Information

- A simpler model comes when we assume there is only one possible state of the world, and there each player knows all other players past moves, $S_i = \times_{j \neq i} D_j$.
- Such a game is called a repeated game with standard information.
- A standard repeated game is consist of a stage game that is played again and again.
- Any strategy that leads to a NE in every stage game is a NE in the repeated game.

Dynamic Games of Complete Information

- In Forwarder's Dilemma we saw there was only one NE which was inefficient.
- Let as assume each time they play there is a 0.99 chance that they will play again.
- The number of times they will play is a random variable with geometric distribution.
- The probability of playing for exactly k rounds is $0.99^{k-1}.01$.
- Suppose that both players play F until one of them decides to play D, and in that case both play D from then on.

Dynamic Games of Complete Information

- The total expected future payoff for both players ass long as they play F is

$$\sum_{k=1}^{\infty}(0.99)^{k-1}(0.01)(1-c)k = 100(1-c)$$

- if player i chooses D at some round then his total expected future payoff will be

$$1+\sum_{k=1}^{\infty}(0.99)^{k-1}(0.01)0k = 1$$

- The strategy of always forwarding is a NE in the reapeted game.
- In fact its a subgame perfect equilibrium because (D,D) is a NE of the stage game.

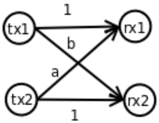
Power Control In Cellular Systems

- The problem of power control has often been modeled as a game.
- Utilities are chosen to be increasing in SINR and decreasing with power.
- One possible utility function used is

$$u_i(\mathbf{p}) = u_i(p_i, \gamma_i) = \frac{R}{p_i}(1 - 2BER(\gamma_i))^L$$

- R is the rate at which user transmits.
- It has been shown that the static game has a unique NE.
- The NE of the static game however is Pareto inefficient.
- If players played the game repeatedly they could enforce better cooperation using credible threats.

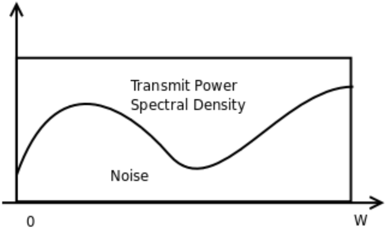
Iterative Waterfilling



- suppose we have the same model as in Random Access Game but now players are allowed to choose their transmitting power spectral densities.
- The two transmitters are the players.
- Each player's strategies are its available power spectral densities.
- players utilities are the rates obtained when they see the other player's signal as interference.

Iterative Waterfilling

- $R_1 = \int_0^W log\left(1 + \frac{p_1(f)}{a(f)p_2(f) + N_0 W}\right)$,
  $R_2 = \int_0^W log\left(1 + \frac{p_2(f)}{b(f)p_1(f) + N_0 W}\right)$
- It can be shown that the optimum transmit signal power spectral density is a waterfilling solution to power spectral density of the noise.



Iterative Waterfilling

- In our model the power spectral density of the noise depends on the transmit power spectral density of the other player.
- When one player changes his PSD the other will also have to change and so on.
- Will they converge to some stable PSDs if they change their power spectral densities without any coordination (in a distributed manner that is)?
- It can be shown that a distributed iterative algorithm where each player does waterfilling to the PSD of the noise plus interference always convergence to a NE.
- This means that players in a game can sometimes converge to an equilibrium distributedly.