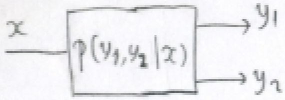
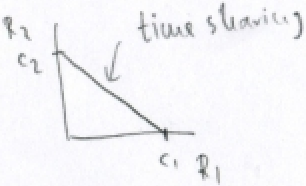


$p(y_1|x) = \sum_{y_2} p(y_1,y_2|x)$, $p(y_2|x) = \sum_{y_1} p(y_1,y_2|x)$

j-th component channel: $X \to Y_j$

Capacity of j-th channel: $C_j = \max_{p(x)} I(X;Y_j)$



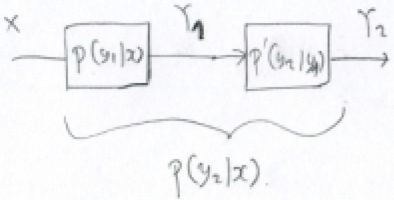
Achievable rates.

- Time sharing: $\tau_1, \tau_2 \ge 0$, $\tau_1 + \tau_2 = 1$.

  $R = (\tau_1 C_1, \tau_2 C_2)$ is achievable by simple time-sharing.

- physically degraded bc: $p(y_1,y_2|x) = p(y_1|x)\cdot p(y_2|y_1)$ $\quad X \leftrightarrow Y_1 \leftrightarrow Y_2$ (Subcase of stoch. degr.: $\overbrace{\sum_{y_1} p(y_1,y_2,x)}^{p(y_2|x)} = \sum_{y_1} p(y_1|x)\cdot p(y_2|y_1)$)
- stochastically degraded: if its conditional marginals same as that of the corresponding physically degraded,

  That is, $\exists\, p'(y_2|y_1)$ such that $p(y_2|x) = \sum_{y_1} p(y_1|x)\cdot p'(y_2|y_1)$



Theorem 14.6.1:

The capacity region of a bc depends only on conditional marginals $p(y_1|x)$, $p(y_2|x)$.
Thus, ANY 2 b.c. with the same conditional marginals have the SAME capacity region.

Proof: $P_1^{(n)} = P\{\hat{W}_1(Y_1) \neq W_1\}$, $P_2^{(n)} = P\{\hat{W}_2(Y_2) \neq W_2\}$

$$\max(P_1^{(n)}, P_2^{(n)}) \leq P_{bc}^{(n)} = P\{(\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)\} \leq P_1^{(n)} + P_2^{(n)}$$

$P_{bc}^{(n)} \to 0$ iff $P_1^{(n)} \to 0$ and $P_2^{(n)} \to 0$.

$P_{bc}^{(n)}$ depends on $p(y_1, y_2|x)$

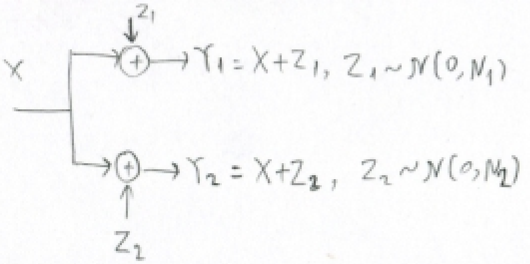
But $P_1^{(n)}$ depends only on $p(y_1|x)$

$P_2^{(n)}$ " " $p(y_2|x)$

Thus, if a sequence of codes for a b.c. gives $P_{bc}^{(n)} \to 0$ ($\Rightarrow P_1^{(n)} \to 0$ and $P_2^{(n)} \to 0$) then the same sequence of codes for another b.c. with SAME MARGINALS will lead to $P_1^{(n)} \to 0$ and $P_2^{(n)} \to 0$ ($\Rightarrow P_{bc'}^{(n)} \to 0$)

Broadcast Gaussian channel.

(A)



Z_1, Z_2 independent

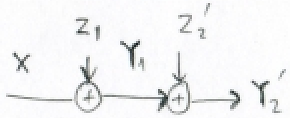
$N_2 > N_1$

- Test for physical degradedness

$$f(y_1, y_2|x) \overset{?}{=} f(y_1|x) \cdot f(y_2|y_1)$$

- Existence of a stochastically degraded version.

(B)



$Z_1 \sim \mathcal{N}(0, N_1)$, $Z_2' \sim \mathcal{N}(0, N_2 - N_1)$

Z_1, Z_2' independent

$$f(y_2'|x) = f(y_2|x)$$

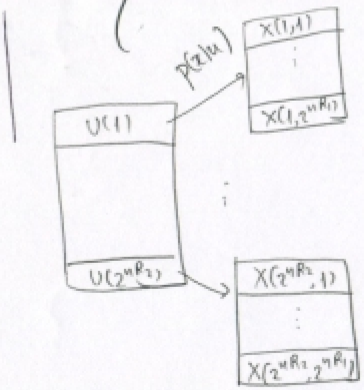
Of course Z_1 and Y_1 are the same in both cases.

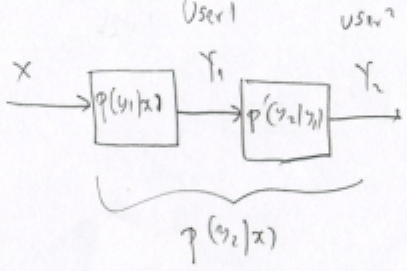
Thus, from a CAPACITY region point of view, systems A and B are equivalent!!!

Ορισμός: $P_1^{(n)} = P\{\hat{W}_1(Y_1) \neq W_1\} = \sum_{x^n} p(x^n) \sum_{y^n} p(y_1^n|x^n) \, I(\hat{W}_1(y_1) \neq W_1)$

Info Theory

PTO codes → clouds, satellites





If User 1 uses only $X(i,j)$ and not $U(i)$, the probability of error is

$$P_1^{(n)} = P\left\{ (X(1,1), Y_1) \notin A_\epsilon^{(n)} \cup \bigcup_{i=1}^{2^{nR_2}} \bigcup_{\substack{j=2 \\ (i,j)\neq(1,1)}}^{2^{nR_1}} (X(i,j), Y_1) \in A_\epsilon^{(n)} \right\}$$

$$\leq \epsilon + 2^{n(R_1+R_2)}\, 2^{-nI(X;Y_1)} \leq 2\epsilon \quad \text{if} \quad R_1+R_2 < I(X;Y_1).$$

Thus, we have

$$\boxed{\begin{aligned} R_2 &< I(U;Y_2) \\ R_1+R_2 &< I(X;Y_1) \end{aligned}}$$

Using $U(i)$ and $X(\hat{i},j)$ we obtain

$$R_2 < I(U;Y_2) \leq I(U;Y_1)$$

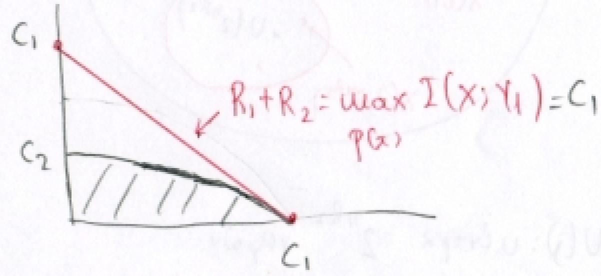
$$R_1 < I(X;Y_1|U)$$

$$R_1+R_2 < I(U;Y_1) + I(X;Y_1|U) = I(X,U;Y_1) = I(X;Y_1) + I(U;Y_1|X) \geq I(X;Y_1)$$

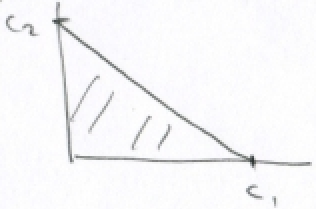
③

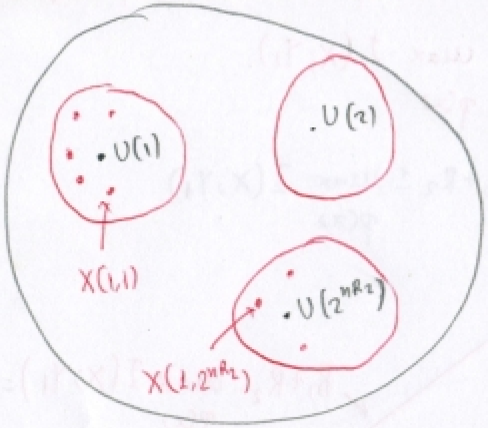
Ερώτημα: Τι σχέση έχει το R_1+R_2 με το $\max_{p(x)} I(X;Y_1)$.

Απάντηση: $R_1+R_2 \leq \max_{p(x)} I(X;Y_1)$



- Αν $C_1 = C_2$, τότε η περιοχή ρυθμών είναι αυτή που επιτυγχάνεται από το time-sharing





- $U(j)$: κέντρα 2^{nR_2} νεφών

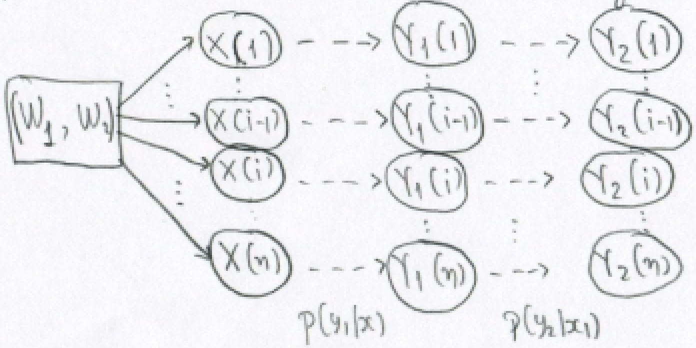
- $X(i,j)$: δορυφόροι του $U(j)$

Converse: Αν $\exists\ ((2^{nR_1}, 2^{nR_2}), n)$-ακολουθία κωδίκων με $P_e^{(n)} \to 0$, τότε $R_1 \le I(X;Y_1|U)$, $R_2 \le I(U;Y_2)$ για

$$\boxed{\begin{array}{c} p(u)\,p(x|u)\,p(y_1|x)\cdot p(y_2|y_1) \\ U \to X \to Y_1 \to Y_2 \end{array}}$$

⊛

Dependency graph of random variables appearing in degraded broadcast scenario.



Παρατήρηση 1: Υπό συνθήκη Y_1^{i-1}, το Y_2^{i-1} είναι ανεξάρτητο από ΟΛΕΣ τις άλλες τυχαίες μεταβλητές.

" 2: Έστω $U_i = (W_2, Y_1^{i-1})$, $i = 1, \dots, n$.

" 3: $U_i \to X_i \to Y_{1i} \to Y_{2i}$. Άρα, ικανοποιείται ο περιορισμός της pmf του κώδικα.

" 4: $Y_{1i} \perp (U_i, W_1) \mid X_i$. Υπό συνθήκη X_i, το Y_{1i} είναι ανεξάρτητο όλων των άλλων τυχ. μεταβλητών.

---

① **Fano:** $H(W_2|Y_2^n) \le n\epsilon_n$, $\epsilon_n \to 0$ as $n \to \infty$

$$nR_2 = H(W_2) = I(W_2;Y_2^n) + H(W|Y_2^n) \le I(W_2;Y_2^n) + n\epsilon_n = \sum_{i=1}^{n} I(W_2;Y_{2i}|Y_2^{i-1}) + n\epsilon_n = \sum_{i=1}^{n}\left[H(Y_{2i}|Y_2^{i-1}) - H(Y_{2i}|Y_2^{i-1},W_2)\right] + n\epsilon_n$$

$$\le \sum_{i=1}^{n}\left[H(Y_{2i}) - H(Y_{2i}|Y_2^{i-1},W_2,Y_1^{i-1})\right] + n\epsilon_n$$ διότι $H(Y_{2i}|Y_2^{i-1}) \le H(Y_{2i})$, $H(Y_{2i}|Y_2^{i-1},W_2) \ge H(Y_{2i}|Y_2^{i-1},W_2,Y_1^{i-1})$

$$= \sum_{i=1}^{n}\left[H(Y_{2i}) - \left(I(Y_{2i};Y_2^{i-1}|W_2,Y_1^{i-1}) + H(Y_{2i}|W_2,Y_1^{i-1})\right)\right] + n\epsilon_n$$ εξαιτίας του ορισμού αμοιβαίας πληροφορίας

$$= \sum_{i=1}^{n}\left[H(Y_{2i}) - H(Y_{2i}|W_2,Y_1^{i-1})\right] + n\epsilon_n$$ εξαιτίας της Παρατήρησης 1: $I(Y_{2i};Y_2^{i-1}|W_2,Y_1^{i-1}) = 0$

$$= \sum_{i=1}^{n}\left[H(Y_{2i}) - H(Y_{2i}|U_i)\right] + n\epsilon_n$$ εξαιτίας του ορισμού του U_i.

② Fano: $H(W_1|Y_1^n) \le n\epsilon_n$, $\epsilon_n \to 0$ as $n \to \infty$.

$$nR_1 = H(W_1) = I(W_1;Y_1^n) + H(W_1|Y_1^n) \le I(W_1;Y_1^n) + n\epsilon_n \qquad \text{Fano}$$

$$\le I(W_1;Y_1^n,W_2) + n\epsilon_n \qquad I(X;Y,Z) \ge I(X;Y)$$

$$= \underbrace{I(W_1;W_2)}_{0} + I(W_1;Y_1^n|W_2) + n\epsilon_n \qquad W_1, W_2 \text{ ανεξάρτητα}$$

$$= \sum_{i=1}^{n} I(W_1;Y_{1i}|W_2,Y_1^{i-1}) + n\epsilon_n = \sum_{i=1}^{n} I(W_1;Y_{1i}|U_i) + n\epsilon_n \qquad \text{εξαιτίας ορισμού } U_i$$

$$= \sum_{i=1}^{n} \left[H(Y_{1i}|U_i) - H(Y_{1i}|U_i,W_1)\right] + n\epsilon_n$$

$$\le \sum_{i=1}^{n} \left[H(Y_{1i}|U_i) - H(Y_{1i}|U_i,W_1,X_i)\right] + n\epsilon_n \qquad \text{διότι } H(Y_{1i}|U_i,W_1) \ge H(Y_{1i}|U_i,W_1,X_i)$$

$$= \sum_{i=1}^{n} \left[H(Y_{1i}|U_i) - (-\underbrace{I(Y_{1i};W_1|U_i,X_i)}_{0} + H(Y_{1i}|U_i,X_i))\right] + n\epsilon_n \qquad \text{διότι } I(Y_{1i};W_1|U_i,X_i) = H(Y_{1i}|U_i,X_i) - H(Y_{1i}|U_i,X_i,W_1)$$

δεδομένου του X_i, το Y_{1i} είναι ανεξάρτητο όλων των άλλων τ.μ.

$$= \sum_{i=1}^{n} \left[H(Y_{1i}|U_i) - H(Y_{1i}|X_i,U_i)\right] + n\epsilon_n$$

$$= \sum_{i=1}^{n} I(Y_{1i};X_i|U_i) + n\epsilon_n$$

Άρα

$$\left.\begin{aligned} R_2 &\le \frac{1}{n}\sum_{i=1}^{n} I(U_i;Y_{2i}) + \epsilon_n \\ R_1 &\le \frac{1}{n}\sum_{i=1}^{n} I(X_i;Y_{1i}|U_i) + \epsilon_n \end{aligned}\right\} \Rightarrow \underline{R} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \le \sum_{i=1}^{n} \frac{1}{n} \underline{I}_i = \lambda_1 \underline{I}_{i_1} + \lambda_2 \underline{I}_{i_2} + \lambda_3 \underline{I}_{i_3}$$

Converse of degraded broadcast (cont.).

Using a time-sharing variable Q, $p(Q=i)=\frac{1}{n}$

$$R_2 \le I(U_Q ; Y_{2Q} \mid Q)$$

$$R_1 \le I(X_{1Q} ; Y_{1Q} \mid U_Q, Q)$$

for some $p(q) \cdot p(u|q)\, p(x|u,q)\, p(y_1,y_2|x)$.