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Control of 2×2 linear hyperbolic systems: Backstepping-based trajectory generation and PI-based tracking[☆]

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ABSTRACT

We consider the problems of trajectory generation and tracking for general 2×2 systems of first-order linear hyperbolic PDEs with anti-collocated boundary input and output. We solve the trajectory generation problem via backstepping. The reference input, which generates the desired output, incorporates integral operators acting on advanced and delayed versions of the reference output with kernels which were derived by Vazquez, Krstic, and Coron for the backstepping stabilization of 2×2 linear hyperbolic systems. We apply our approach to a wave PDE with indefinite in-domain and boundary damping. For tracking the desired trajectory we employ a PI control law on the tracking error of the output. We prove exponential stability of the closed-loop system, under the proposed PI control law, when the parameters of the plant and the controller satisfy certain conditions, by constructing a novel "non-diagonal" Lyapunov functional. We demonstrate that the proposed PI control law compensates in the output the effect of in-domain and boundary disturbances. We illustrate our results with numerical examples.

1. Introduction

Control of 2×2 systems of first-order hyperbolic PDEs is an active area of research since numerous processes can be modeled with this class of PDE systems. Among various applications, 2×2 systems model the dynamics of traffic [1,2], hydraulic [3–6], as well as gas pipeline networks [7], and the dynamics of transmission lines [8].

Several articles are dedicated to the control and analysis of 2×2 linear [3,9,5,10] [11–13] and nonlinear [14–19] systems. Results for the control of $n \times n$ systems also exist [20–23]. Algorithms for disturbance rejection in 2×2 systems are recently developed [24,25]. The motion planning problem is solved in [26,27], for a class of 2×2 systems and in [28,29] for a class of wave PDEs. Perhaps the most relevant results to the present article are the results in [5], dealing with the Lyapunov-based output-feedback control of 2×2 linear systems, the results in [12], dealing with the backstepping stabilization of 2×2 linear systems, and the results in [27], dealing with the motion planning for a class of 2×2 systems.

In this paper, we are concerned with the trajectory generation and tracking problems for general 2×2 systems of first-order linear hyperbolic PDEs with anti-collocated boundary input and output. We solve the motion planning problem for this class of systems employing backstepping (Section 2.1). Specifically, we start from a simple transformed system, namely, a cascade of two first-order hyperbolic PDEs, for which the motion planning problem can be trivially solved. We then apply an inverse backstepping transformation to derive the reference trajectory and reference input for the original system. Our approach is different than the one in [12], in that we use backstepping for trajectory generation rather than stabilization, and the one in [27], in that we employ a different conceptual idea to a different class of systems. The idea of the backstepping-based trajectory generation for PDEs, which was conceived in [30], is applied to a beam PDE in [31] and the Navier–Stokes equations in [32], and is recently extended to general $n \times n$ linear hyperbolic systems in [22]. We apply this methodology to a wave PDE with indefinite in-domain and boundary damping by transforming (see, for example, [33]) the wave PDE to a 2×2 linear hyperbolic system coupled with a first-order ODE (Section 2.2).

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We then employ a PI control law for the stabilization of the error system, namely, the system whose state is defined as the difference between the state of the plant and the reference trajectory. We prove exponential stability in the L_2 norm of the closed-loop system by constructing a Lyapunov functional which incorporates cross-terms between the PDE states of the system and the ODE state of the controller, when the parameters of the system and the controller satisfy certain conditions (Section 3.1). Our result differs than the result in [5] in that we employ PI control on an output of the system in the Riemann coordinates and we construct a non-diagonal Lyapunov functional for proving closed-loop stability. We demonstrate that the proposed PI control law is capable of compensating in the output the effect of additive disturbances affecting the boundary or the interior of the PDE domain (Section 3.2). We present several examples, for the illustration of our methodologies, including a simulation example dealing with the generation of a sinusoidal reference trajectory for a wave PDE (Section 4.1) and a simulation example of a system tracking a sinusoidal reference output (Section 4.2).

2. Trajectory generation using backstepping

2.1. General 2 × 2 linear hyperbolic systems

We consider the following system

$$z^1_t + \varepsilon_1(x)z^1_x = c_1(x)z^1 + c_2(x)z^2 \tag{1}$$

$$z^2_t - \varepsilon_2(x)z^2_x = c_3(x)z^1 + c_4(x)z^2, \tag{2}$$

under the boundary conditions

$$z^1(0,t) = qz^2(0,t) \tag{3}$$

$$z^2(1,t) = S(t) \tag{4}$$

$$z^2(0,t) = y(t), \tag{5}$$

where $t \in [0,+\infty)$ is the time variable, $x \in [0,1]$ is the spatial variable, y is the output of the system, and S is the control input. The functions $\varepsilon_1, \varepsilon_2$ belong to $C^2([0,1])$ and satisfy $\varepsilon_1(x), \varepsilon_2(x) > 0$, for all $x \in [0,1]$, and the functions c_i, $i = 1,2,3,4$ belong to $C^1([0,1])$.

Defining the change of variables (see, for example, [3])

$$\chi_1(x) = \exp\left(-\int_0^x \frac{c_1(s)}{\varepsilon_1(s)}ds\right) \tag{6}$$

$$\chi_2(x) = \exp\left(\int_0^x \frac{c_4(s)}{\varepsilon_2(s)}ds\right) \tag{7}$$

$$\chi(x) = \frac{\chi_1(x)}{\chi_2(x)}, \tag{8}$$

and the new coordinates

$$u = \chi_1(x)z^1 \tag{9}$$

$$v = \chi_2(x)z^2, \tag{10}$$

system (1)–(5) is transformed into the following system

$$u_t + \varepsilon_1(x)u_x = \gamma_1(x)v \tag{11}$$

$$v_t - \varepsilon_2(x)v_x = \gamma_2(x)u, \tag{12}$$

with

$$\gamma_1(x) = \chi(x)c_2(x) \tag{13}$$

$$\gamma_2(x) = \chi^{-1}(x)c_3(x). \tag{14}$$

The boundary conditions become

$$u(0,t) = qv(0,t) \tag{15}$$

$$v(1,t) = U(t) \tag{16}$$

$$v(0,t) = y(t), \tag{17}$$

where the original control variable satisfies

$$U = \chi_2(1)S. \tag{18}$$

We aim at designing a reference control input $U^r(t)$ such that the output $y(t)$ follows a given reference trajectory $y^r(t)$, for $t \geq 0$. For achieving this we need first to construct the reference trajectory $(u^r(x,t), v^r(x,t))$ that satisfies (11), (12), (15), and (17) with $y(t) = y^r(t)$. The trajectory generation problem is solvable when the initial data (u^0, v^0) match the reference trajectory, i.e., when $u^0(x) = u^r(x,0)$ and $v^0(x) = v^r(x,0)$ (and hence, the initial data belong to the same space with $u^r(x,0)$ and $v^r(x,0)$).

Theorem 1. *Let $y^r \in C^1(\mathbb{R})$ be uniformly bounded. The functions*

$$\begin{aligned} u^r(x,t) = {}& qy^r(t-\Phi_1(x)) + \int_0^x \frac{f(\xi)}{\varepsilon_1(\xi)} y^r(t-\Phi_1(x)+\Phi_1(\xi))\,d\xi \\ &+ q\int_0^x L^{\alpha\alpha}(x,\xi)y^r(t-\Phi_1(\xi))d\xi \\ &+ \int_0^x L^{\alpha\alpha}(x,\xi)\int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^r(t-\Phi_1(\xi)+\Phi_1(\zeta))\,d\zeta\,d\xi \\ &+ \int_0^x L^{\alpha\beta}(x,\xi)\,y^r(t+\Phi_2(\xi))\,d\xi \end{aligned} \tag{19}$$

$$\begin{aligned} v^r(x,t) = {}& y^r(t+\Phi_2(x)) + q\int_0^x L^{\beta\alpha}(x,\xi)y^r(t-\Phi_1(\xi))\,d\xi \\ &+ \int_0^x L^{\beta\alpha}(x,\xi)\int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^r(t-\Phi_1(\xi)+\Phi_1(\zeta))\,d\zeta\,d\xi \\ &+ \int_0^x L^{\beta\beta}(x,\xi)y^r(t+\Phi_2(\xi))\,d\xi \end{aligned} \tag{20}$$

$$\begin{aligned} U^r(t) = {}& y^r(t+\Phi_2(1)) + q\int_0^1 L^{\beta\alpha}(1,\xi)y^r(t-\Phi_1(\xi))\,d\xi \\ &+ \int_0^1 L^{\beta\alpha}(1,\xi)\int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^r(t-\Phi_1(\xi)+\Phi_1(\zeta))\,d\zeta\,d\xi \\ &+ \int_0^1 L^{\beta\beta}(1,\xi)y^r(t+\Phi_2(\xi))\,d\xi, \end{aligned} \tag{21}$$

where

$$\Phi_1(x) = \int_0^x \frac{1}{\varepsilon_1(s)}ds \tag{22}$$

$$\Phi_2(x) = \int_0^x \frac{1}{\varepsilon_2(s)}ds \tag{23}$$

$$f(x) = \begin{cases} \varepsilon_2(0)K^{uv}(x,0), & \text{if } q = 0 \\ 0, & \text{if } q \neq 0, \end{cases} \tag{24}$$

and $L^{\alpha\alpha}, L^{\alpha\beta}, L^{\beta\alpha}, L^{\beta\beta}, K^{uv}$ are the solutions of the following equations

$$\varepsilon_2(x)L^{\beta\alpha}_x - \varepsilon_1(\xi)L^{\beta\alpha}_\xi = \varepsilon_1'(\xi)L^{\beta\alpha} - \gamma_2(x)L^{\alpha\alpha} \tag{25}$$

$$\varepsilon_2(x)L^{\beta\beta}_x + \varepsilon_2(\xi)L^{\beta\beta}_\xi = -\varepsilon_2'(\xi)L^{\beta\beta} - \gamma_2(x)L^{\alpha\beta} \tag{26}$$

$$\varepsilon_1(x)L^{\alpha\alpha}_x + \varepsilon_1(\xi)L^{\alpha\alpha}_\xi = -\varepsilon_1'(\xi)L^{\alpha\alpha} + \gamma_1(x)L^{\beta\alpha} \tag{27}$$

$$\varepsilon_1(x)L^{\alpha\beta}_x - \varepsilon_2(\xi)L^{\alpha\beta}_\xi = \varepsilon_2'(\xi)L^{\alpha\beta} + \gamma_1(x)L^{\beta\beta} \tag{28}$$

$$\varepsilon_1(x)K^{uu}_x + \varepsilon_1(\xi)K^{uu}_\xi = -\varepsilon_1'(\xi)K^{uu} - \gamma_2(x)K^{uv} \tag{29}$$

$$\varepsilon_1(x)K^{uv}_x - \varepsilon_2(\xi)K^{uv}_\xi = \varepsilon_2'(\xi)K^{uv} - \gamma_1(x)K^{uu}, \tag{30}$$

with the boundary conditions

$$L^{\beta\alpha}(x,x) = -\frac{\gamma_2(x)}{\varepsilon_1(x)+\varepsilon_2(x)} \tag{31}$$

$$L^{\alpha\alpha}(x,0) = \begin{cases} h_1(x), & \textit{if } q = 0 \\ \dfrac{\varepsilon_2(0)}{q\varepsilon_1(0)} L^{\alpha\beta}(x,0), & \textit{if } q \neq 0 \end{cases} \tag{32}$$

$$L^{\beta\beta}(x,0) = \begin{cases} \dfrac{1}{\varepsilon_2(0)} \displaystyle\int_0^x L^{\beta\alpha}(x,\xi) f(\xi) d\xi, & \textit{if } q = 0 \\ \dfrac{q\varepsilon_1(0)}{\varepsilon_2(0)} L^{\beta\alpha}(x,0), & \textit{if } q \neq 0 \end{cases} \tag{33}$$

$$L^{\alpha\beta}(x,x) = \frac{\gamma_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)} \tag{34}$$

$$K^{uu}(x,0) = h_2(x) \tag{35}$$

$$K^{uv}(x,x) = \frac{\gamma_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)}, \tag{36}$$

where h_1, $h_2 \in C^1([0,1])$ are arbitrary, uniformly bounded and solve the boundary value problem (11), (12), (15), (16). In particular, $v^r(0,t) = y^r(t)$, for $t \geq 0$.

Before proving Theorem 1 we make the following observation, which is also helpful in understanding better the proof strategy of Theorem 1.

Remark 1. The approach for the trajectory generation introduced here is inspired from backstepping. Consider the following system

$$\alpha_t + \varepsilon_1(x)\alpha_x - f(x)\beta(0,t) = 0 \tag{37}$$

$$\beta_t - \varepsilon_2(x)\beta_x = 0, \tag{38}$$

with boundary condition

$$\alpha(0,t) = q\beta(0,t), \tag{39}$$

which follows by directly applying the backstepping transformation

$$\alpha(x,t) = u(x,t) - \int_0^x K^{uu}(x,\xi)\, u(\xi,t)\, d\xi - \int_0^x K^{uv}(x,\xi)\, v(\xi,t)\, d\xi \tag{40}$$

$$\beta(x,t) = v(x,t) - \int_0^x K^{vu}(x,\xi)\, u(\xi,t)\, d\xi - \int_0^x K^{vv}(x,\xi)\, v(\xi,t)\, d\xi, \tag{41}$$

where the kernels K^{uu}, K^{uv}, K^{vu}, K^{vv} are given in [12], to system (11), (12), and (15). It is shown that the functions

$$\alpha(x,t) = qy^r(t-\Phi_1(x)) + \int_0^x \frac{f(\xi)}{\varepsilon_1(\xi)} y^r(t-\Phi_1(x)+\Phi_1(\xi))\, d\xi \tag{42}$$

$$\beta(x,t) = y^r(t+\Phi_2(x)), \tag{43}$$

where Φ_1 and Φ_2 are defined in (22) and (23), respectively, satisfy (37)–(39) with

$$\beta(1,t) = y^r(t+\Phi_2(1)) \tag{44}$$

and, in particular, $\beta(0,t) = y^r(t)$. Using the inverse backstepping transformations introduced in [12]

$$u(x,t) = \alpha(x,t) + \int_0^x L^{\alpha\alpha}(x,\xi)\alpha(\xi,t)d\xi + \int_0^x L^{\alpha\beta}(x,\xi)\beta(\xi,t)d\xi \tag{45}$$

$$v(x,t) = \beta(x,t) + \int_0^x L^{\beta\alpha}(x,\xi)\alpha(\xi,t)d\xi + \int_0^x L^{\beta\beta}(x,\xi)\beta(\xi,t)d\xi, \tag{46}$$

and relations (42), (43) one can conclude that the functions u^r, v^r, and $U^r = v^r(1)$ solve the trajectory generation problem for system (11), (12), (15)–(17).

Note that the present approach cannot be directly applied to cases where $\varepsilon_1(x)$ or $\varepsilon_2(x)$ vanish for some $x \in [0,1]$. This is evident, for instance, from (33) which would imply that the kernel $L^{\beta\beta}$ of the open-loop control law U^r may become infinity for all $x \in [0,1]$.

Proof. We first consider the case $q \neq 0$. Note that since ε_1, $\varepsilon_2 \in C^2([0,1])$ with $\varepsilon_1(x), \varepsilon_2(x) > 0$, for all $x \in [0,1]$ and γ_1, $\gamma_2 \in C^1([0,1])$, system (25)–(34) has a unique solution with $L^{\alpha\alpha}$, $L^{\alpha\beta}$, $L^{\beta\alpha}$, $L^{\beta\beta} \in C^1(\mathcal{T})$ where $\mathcal{T} = \{(x,\xi) : 0 \leq \xi \leq x \leq 1\}$ [16]. Hence, from (19)–(21) and the uniform boundedness of y^r it follows that u^r, v^r, and U^r are bounded for all $t \geq 0$ and $x \in [0,1]$.

Taking the time and space derivatives of u^r we get

$$\begin{aligned} u_t^r + \varepsilon_1(x)u_x^r &= q\int_0^x L^{\alpha\alpha}(x,\xi) y^{r\prime}(t-\Phi_1(\xi))d\xi \\ &+ \int_0^x L^{\alpha\beta}(x,\xi)y^{r\prime}(t+\Phi_2(\xi))d\xi \\ &+ \varepsilon_1(x)\int_0^x L_x^{\alpha\beta}(x,\xi)y^r(t+\Phi_2(\xi))\,d\xi \\ &+ q\varepsilon_1(x)\int_0^x L_x^{\alpha\alpha}(x,\xi)y^r(t-\Phi_1(\xi))\,d\xi \\ &+ \varepsilon_1(x)L^{\alpha\beta}(x,x)y^r(t+\Phi_2(x)) \\ &+ q\varepsilon_1(x)L^{\alpha\alpha}(x,x)y^r(t-\Phi_1(x)). \end{aligned} \tag{47}$$

Integrating by parts the first two integrals we get

$$\begin{aligned} u_t^r + \varepsilon_1(x)u_x^r &= q\int_0^x \Big(\varepsilon_1(x)L_x^{\alpha\alpha}(x,\xi) \\ &+ \varepsilon_1(\xi)L_\xi^{\alpha\alpha}(x,\xi) + \varepsilon_1'(\xi)L^{\alpha\alpha}(x,\xi)\Big) y^r(t-\Phi_1(\xi))d\xi \\ &+ \int_0^x \Big(\varepsilon_1(x)L_x^{\alpha\beta}(x,\xi) - \varepsilon_2(\xi)\, L_\xi^{\alpha\beta}(x,\xi) \\ &- \varepsilon_2'(\xi)\, L^{\alpha\beta}(x,\xi)\Big) y^r(t+\Phi_2(\xi))d\xi \\ &+ \big(q\varepsilon_1(0)L^{\alpha\alpha}(x,0) - \varepsilon_2(0)L^{\alpha\beta}(x,0)\big) y^r(t) \\ &+ (\varepsilon_1(x)+\varepsilon_2(x))\, L^{\alpha\beta}(x,x)y^r(t+\Phi_2(x)). \end{aligned} \tag{48}$$

Due to the fact that $L^{\alpha\beta}$ and $L^{\alpha\alpha}$ are the solutions of (27) and (28) with the boundary conditions (32) and (34) one gets, by using (20), that u^r satisfies (11). The proof that v^r satisfies (12) follows analogously. Setting $x = 0$ in (19), (20) and using (22), (23), we get that u^r and v^r satisfy (15). Setting $x = 1$ in (20) it follows that (21) satisfies (16). Setting in (20) $x = 0$ and using (23) we get $v^r(0,t) = y^r(t)$.

Let us consider next the case $q = 0$. First observe that the PDEs (25), (27) with boundary conditions (31), (32), for the kernels $L^{\alpha\alpha}$ and $L^{\beta\alpha}$ are decoupled, and hence, $L^{\alpha\alpha}$ and $L^{\beta\alpha}$ are well-defined [16]. Hence, since f satisfies (24) and K^{uv}, K^{uu} are well-defined [16], one can conclude that $L^{\alpha\beta}$ and $L^{\beta\beta}$ are well-defined as well.

Taking the time and space derivatives of u^r we get

$$u_t^r + \varepsilon_1(x)u_x^r = f(x)y^r(t) + \int_0^x L^{\alpha\alpha}(x,\xi) \times \int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^{r\prime}(t-\Phi_1(\xi)+\Phi_1(\zeta))\,d\zeta\, d\xi$$

$$+\int_0^x L^{\alpha\beta}(x,\xi)y^{r\prime}(t+\Phi_2(\xi))d\xi$$
$$+\varepsilon_1(x)L^{\alpha\alpha}(x,x)\int_0^x \frac{f(\zeta)}{\varepsilon_1(\zeta)}y^r(t-\Phi_1(x)+\Phi_1(\zeta))d\zeta$$
$$+\varepsilon_1(x)L^{\alpha\beta}(x,x)y^r(t+\Phi_2(x))$$
$$+\varepsilon_1(x)\int_0^x L_x^{\alpha\beta}(x,\xi)y^r(t+\Phi_2(\xi))d\xi$$
$$+\varepsilon_1(x)\int_0^x L_x^{\alpha\alpha}(x,\xi)\int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)}y^r(t-\Phi_1(\xi)$$
$$+\Phi_1(\zeta))d\zeta d\xi. \tag{49}$$

Integrating by parts the first two integrals we get

$$u_t^r+\varepsilon_1(x)u_x^r$$
$$=\int_0^x\left(\varepsilon_1(x)L_x^{\alpha\alpha}(x,\xi)+\varepsilon_1(\xi)L_\xi^{\alpha\alpha}(x,\xi)+\varepsilon_1'(\xi)L^{\alpha\alpha}(x,\xi)\right)$$
$$\times\int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)}y^r(t-\Phi_1(\xi)+\Phi_1(\zeta))d\zeta d\xi$$
$$+\int_0^x\Big(\varepsilon_1(x)L_x^{\alpha\beta}(x,\xi)$$
$$-\ \varepsilon_2(\xi)L_\xi^{\alpha\beta}(x,\xi)-\varepsilon_2'(\xi)L^{\alpha\beta}(x,\xi)\Big)y^r(t+\Phi_2(\xi))d\xi$$
$$+\left(\varepsilon_1(x)+\varepsilon_2(x)\right)L^{\alpha\beta}(x,x)y^r(t+\Phi_2(x))$$
$$+y^r(t)\left(f(x)+\int_0^x L^{\alpha\alpha}(x,\xi)f(\xi)d\xi-\varepsilon_2(0)L^{\alpha\beta}(x,0)\right). \tag{50}$$

Using (20), (27), (28), and (34) one can conclude that u^r satisfies (11) if f satisfies

$$f(x)=\varepsilon_2(0)L^{\alpha\beta}(x,0)-\int_0^x L^{\alpha\alpha}(x,\xi)f(\xi)d\xi. \tag{51}$$

This fact can been shown as follows. The inverse of the backstepping transformation (40), (41) is uniquely defined and has the form (45), (46) (see, for example, [34]). Hence, substituting (40), (41) in (45), (46) we get

$$\int_0^x\left(K^{uu}(x,\xi)-L^{\alpha\alpha}(x,\xi)\right)u(\xi,t)+\left(K^{uv}(x,\xi)-\ L^{\alpha\beta}(x,\xi)\right)v(\xi,t)d\xi$$
$$+\int_0^x\int_0^\xi\Big(\left(L^{\alpha\alpha}(x,\xi)K^{uu}(\xi,\zeta)+L^{\alpha\beta}(x,\xi)K^{vu}(\xi,\zeta)\right)u(\zeta,t)$$
$$+\left(L^{\alpha\alpha}(x,\xi)K^{uv}(\xi,\zeta)+L^{\alpha\beta}(x,\xi)K^{vv}(\xi,\zeta)\right)v(\zeta,t)\Big)d\zeta d\xi$$
$$=0. \tag{52}$$

Performing a change in the order of integration in the second integral of (52) and using the fact that (52) holds for all u and v, one obtains

$$K^{uv}(x,\xi)=L^{\alpha\beta}(x,\xi)$$
$$-\int_\xi^x\left(L^{\alpha\alpha}(x,s)K^{uv}(s,\xi)+L^{\alpha\beta}(x,s)K^{vv}(s,\xi)\right)ds. \tag{53}$$

Setting $\xi=0$ in (53), multiplying (53) by $\varepsilon_2(0)$, and using the facts that $K^{vv}(x,0)=0$ for all $x\in[0,1]$ (see relation (31) in [12]) and that f is defined by (24), we get that f satisfies (51) for $q=0$. The rest of the proof is similar to the case $q\neq 0$.

Example 1. We consider the following system

$$z_t^1+\varepsilon_1 z_x^1=-\frac{1}{\tau}z^1 \tag{54}$$
$$z_t^2-\varepsilon_2 z_x^2=-\frac{1}{\tau}z^1, \tag{55}$$

with boundary conditions

$$z^1(0,t)=qz^2(0,t) \tag{56}$$
$$z^2(1,t)=S(t), \tag{57}$$

where τ is a positive parameter. Among various systems that can be modeled by (54)–(57) (for instance, the Saint-Venant equations, see [5,4]), system (54)–(57) can be viewed as a linearized version of the Aw–Rascle–Zhang (ARZ) macroscopic model of traffic flow in the Riemann coordinates

$$z^1=w-V'(s^*)s \tag{58}$$
$$z^2=w, \tag{59}$$

where w and s correspond to the velocity and density of the vehicles at time t and location x, respectively. The variable $V(s^*)$ is the nominal velocity of the cars and s^* is the nominal density. The opposite transport velocities in (54), (55) correspond to traffic flow in a congested mode. The parameter $\frac{1}{\tau}$ is an indicator of the convergence rate of the velocity w of the cars to the nominal velocity $V(s)$. For more details the reader is referred to [1]. The boundary condition (56) in the original variables is written as

$$w=\frac{V'(s^*)s}{1-q}. \tag{60}$$

Hence, the boundary condition (56) dictates that there is a static relation, at the entrance of the road, between the density and the velocity similarly to the static relation between the nominal velocity $V(s)$ and the density of the cars in the road. The change of variables (9), (10), (13), and (14) transform system (54)–(57) to

$$u_t+\varepsilon_1 u_x=0 \tag{61}$$
$$v_t-\varepsilon_2 v_x=-\frac{1}{\tau}\exp\left(-\frac{1}{\tau\varepsilon_1}x\right)u \tag{62}$$
$$u(0,t)=qv(0,t) \tag{63}$$
$$v(1,t)=U(t), \tag{64}$$

where $U(t)$ is given by (18). Observing that $\gamma_1=0$, relations (25)–(34) can be solved explicitly as

$$L^{\alpha\alpha}(x,\xi)=0 \tag{65}$$
$$L^{\alpha\beta}(x,\xi)=0 \tag{66}$$
$$L^{\beta\alpha}(x,\xi)=\frac{1}{\tau\left(\varepsilon_1+\varepsilon_2\right)}\exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1 x+\varepsilon_2\xi}{\varepsilon_1+\varepsilon_2}\right)\right) \tag{67}$$
$$L^{\beta\beta}(x,\xi)=\frac{q\varepsilon_1}{\tau\varepsilon_2\left(\varepsilon_1+\varepsilon_2\right)}\exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1 x-\varepsilon_1\xi}{\varepsilon_1+\varepsilon_2}\right)\right). \tag{68}$$

Therefore, for system (54)–(57), the reference input which generates the desired output $y^r(t)$ is

$$S^r(t)=y^r\left(t+\frac{1}{\varepsilon_2}\right)+\frac{q}{\tau\left(\varepsilon_1+\varepsilon_2\right)}$$
$$\times\int_0^1\exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1+\varepsilon_2\xi}{\varepsilon_1+\varepsilon_2}\right)\right)y^r\left(t-\frac{\xi}{\varepsilon_1}\right)d\xi$$
$$+\frac{q\varepsilon_1}{\tau\varepsilon_2\left(\varepsilon_1+\varepsilon_2\right)}$$
$$\times\int_0^1\exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1-\varepsilon_1\xi}{\varepsilon_1+\varepsilon_2}\right)\right)y^r\left(t+\frac{\xi}{\varepsilon_2}\right)d\xi. \tag{69}$$

2.2. Application to a wave PDE with indefinite in-domain and boundary damping

Let us consider system

$$z_{tt} = \varepsilon(x)z_{xx} + h(x)z_t + b(x)z_x \tag{70}$$
$$z_x(0, t) = -gz_t(0, t) \tag{71}$$
$$z_x(1, t) = W(t), \tag{72}$$

with $g \neq \left\{\frac{1}{\sqrt{\varepsilon(0)}}, -\frac{1}{\sqrt{\varepsilon(0)}}\right\}$, $h, b \in C^1([0, 1])$, and $\varepsilon \in C^2([0, 1])$ with $\varepsilon(x) > 0$, for all $x \in [0, 1]$. The objective is $z(0, t)$ to track a reference trajectory, say, $\zeta(t)$, which belongs to $C^2(\mathbb{R})$. Let us define the output of the system as

$$\psi(t) = z(0, t). \tag{73}$$

With the change of variables

$$z^1(x, t) = \frac{1 - \sqrt{\varepsilon(0)}g}{1 + \sqrt{\varepsilon(0)}g}\left(z_t(x, t) - \sqrt{\varepsilon(x)}z_x(x, t)\right) \tag{74}$$
$$z^2(x, t) = z_t(x, t) + \sqrt{\varepsilon(x)}z_x(x, t) \tag{75}$$
$$S(t) = \sqrt{\varepsilon(1)}W(t) + z_t(1, t), \tag{76}$$

system (70)–(72) is rewritten as (1)–(5) where

$$y(t) = \left(1 - \sqrt{\varepsilon(0)g}\right)\dot{\psi}(t) \tag{77}$$
$$\varepsilon_1(x) = \sqrt{\varepsilon(x)} \tag{78}$$
$$\varepsilon_2(x) = \sqrt{\varepsilon(x)} \tag{79}$$
$$q = 1 \tag{80}$$
$$c_1(x) = \frac{h(x)}{2} - \frac{b(x)}{2\sqrt{\varepsilon(x)}} + \frac{\varepsilon'(x)}{4\sqrt{\varepsilon(x)}} \tag{81}$$
$$c_2(x) = mc_4(x) \tag{82}$$
$$c_3(x) = \frac{1}{m}c_1(x) \tag{83}$$
$$c_4(x) = \frac{h(x)}{2} + \frac{b(x)}{2\sqrt{\varepsilon(x)}} - \frac{\varepsilon'(x)}{4\sqrt{\varepsilon(x)}} \tag{84}$$
$$m = \frac{1 - \sqrt{\varepsilon(0)}g}{1 + \sqrt{\varepsilon(0)}g}, \tag{85}$$

together with the integrator $\dot{\psi}(t) = \frac{1}{1-\sqrt{\varepsilon(0)}g}z^2(0, t)$. Applying Theorem 1 we get the following reference input

$$\begin{aligned} W^r(t) = \frac{1}{2\sqrt{\varepsilon(1)}}\Bigg(&\left(\left(1 - \sqrt{\varepsilon(0)}g\right)\right. \\ &\times \exp\left(-\int_0^1 \frac{c_4(s)}{\varepsilon_2(s)}ds\right)\left(\dot{\zeta}\left(t + \Phi_2(1)\right)\right. \\ &+ \int_0^1 L^{\beta\alpha}(1, \xi)\dot{\zeta}\left(t - \Phi_1(\xi)\right)d\xi \\ &\left.+ \int_0^1 L^{\beta\beta}(1, \xi)\dot{\zeta}\left(t + \Phi_2(\xi)\right)d\xi\right) \\ &- \left(1 + \sqrt{\varepsilon(0)}g\right)\exp\left(\int_0^1 \frac{c_1(s)}{\varepsilon_1(s)}ds\right)\left(\dot{\zeta}\left(t - \Phi_1(1)\right)\right. \\ &+ \int_0^1 L^{\alpha\alpha}(x, \xi)\dot{\zeta}\left(t - \Phi_1(\xi)\right)d\xi \\ &\left.\left.+ \int_0^1 L^{\alpha\beta}(x, \xi)\dot{\zeta}\left(t + \Phi_2(\xi)\right)d\xi\right)\right). \end{aligned} \tag{86}$$

3. Trajectory tracking using PI control

3.1. Stability analysis with a non-diagonal Lyapunov functional

For stabilizing the system around the desired trajectory for any initial condition $(u(x, 0), v(x, 0))$, rather than only for $(u(x, 0), v(x, 0)) = (u^r(x, 0), v^r(x, 0))$, we employ a PI-feedback control law. We first write the dynamics of the tracking errors $\tilde{u}(x, t) = u(x, t) - u^r(x, t)$ and $\tilde{v}(x, t) = v(x, t) - v^r(x, t)$ as

$$\tilde{u}_t + \varepsilon_1(x)\tilde{u}_x = \gamma_1(x)\tilde{v} \tag{87}$$
$$\tilde{v}_t - \varepsilon_2(x)\tilde{v}_x = \gamma_2(x)\tilde{u} \tag{88}$$
$$\tilde{u}(0, t) = q\tilde{v}(0, t) \tag{89}$$
$$\tilde{v}(1, t) = \tilde{U}(t), \tag{90}$$

where $\tilde{U} = U - U^r$ and U^r is the reference input generating the desired reference trajectory. We employ the controller

$$\tilde{U}(t) = -k_P\tilde{v}(0, t) - k_I\tilde{\eta}(t), \tag{91}$$

with

$$\dot{\tilde{\eta}}(t) = \tilde{v}(0, t). \tag{92}$$

Theorem 2. *Consider system (87)–(90) together with the control law (91), (92). Let the positive constants μ, β, ρ, γ, ν, κ, and θ be such that the matrices*

$$M_1 = \begin{bmatrix} -q^2 - \beta\left(k_P^2 e^{\mu} - 1\right) - \frac{\kappa\gamma}{2} & -\beta k_P k_I e^{\mu} + \frac{\gamma}{2}\left(e^{\nu}k_P + 1\right) - \frac{\rho}{2} \\ -\beta k_P k_I e^{\mu} + \frac{\gamma}{2}\left(e^{\nu}k_P + 1\right) - \frac{\rho}{2} & -\beta k_I^2 e^{\mu} + \gamma e^{\nu}k_I - \frac{\gamma}{2} \end{bmatrix} \tag{93}$$

$$M_2(x) = \begin{bmatrix} M_{21}(x) & M_{22}(x) \\ M_{23}(x) & M_{24}(x) \end{bmatrix} \tag{94}$$

with

$$M_{21}(x) = \left(\mu - \frac{\theta}{\varepsilon_1(x)}\right)e^{-\mu x} + \frac{\gamma^2}{2\left(\theta\rho - \gamma\right)}\frac{\gamma_2^2(x)}{\varepsilon_2^2(x)}e^{2\nu x} \tag{95}$$

$$\begin{aligned} M_{22}(x) = &-\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x} - \beta\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x} \\ &- \frac{\gamma^2}{2\left(\theta\rho - \gamma\right)}\frac{\gamma_2(x)}{\varepsilon_2(x)}\left(\nu - \frac{\theta}{\varepsilon_2(x)}\right)e^{2\nu x} \end{aligned} \tag{96}$$

$$\begin{aligned} M_{23}(x) = &-\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x} - \beta\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x} \\ &- \frac{\gamma^2}{2\left(\theta\rho - \gamma\right)}\frac{\gamma_2(x)}{\varepsilon_2(x)}\left(\nu - \frac{\theta}{\varepsilon_2(x)}\right)e^{2\nu x} \end{aligned} \tag{97}$$

$$\begin{aligned} M_{24}(x) = &\beta\left(\mu - \frac{\theta}{\varepsilon_2(x)}\right)e^{\mu x} - \frac{\gamma}{2\kappa}\frac{e^{2\nu x}}{\varepsilon_2^2(x)} \\ &+ \frac{\gamma^2}{2\left(\theta\rho - \gamma\right)}\left(\nu - \frac{\theta}{\varepsilon_2(x)}\right)^2 e^{2\nu x}, \end{aligned} \tag{98}$$

are positive semi-definite for all $x \in [0, 1]$, and the inequalities

$$\beta\rho > \frac{\gamma^2 e^{(2\nu-\mu)x}}{2\varepsilon_2(x)}, \quad \forall x \in [0, 1] \tag{99}$$
$$\gamma > \theta\rho, \tag{100}$$

hold. Then, there exist positive constants λ and Ω such that, for all initial conditions satisfying $\left(\tilde{u}^0(x), \tilde{v}^0(x), \tilde{\eta}^0\right) \in L^2(0, 1) \times$

$L^2(0,1)\times\mathbb{R}$, *the following holds for all* $t \geq 0$

$$\int_0^1 \left(\tilde{u}^2(x,t)+\tilde{v}^2(x,t)\right)dx+\tilde{\eta}^2(t) \leq \Omega e^{-\lambda t}\left(\int_0^1 \left(\tilde{u}^2(x,0)+\tilde{v}^2(x,0)\right)dx+\tilde{\eta}^2(0)\right). \quad (101)$$

Proof. In order to analyze the stability of system (87)–(92) we propose the following Lyapunov functional

$$V(t) = \int_0^1 \begin{bmatrix}\tilde{u}(x,t)\\ \tilde{v}(x,t)\\ \tilde{\eta}(t)\end{bmatrix}^\top P(x)\begin{bmatrix}\tilde{u}(x,t)\\ \tilde{v}(x,t)\\ \tilde{\eta}(t)\end{bmatrix}dx = R_1(t)+R_2(t)+R_3(t)+R_4(t), \quad (102)$$

with

$$P(x) = \begin{bmatrix}\frac{e^{-\mu x}}{\varepsilon_1(x)} & 0 & 0\\ 0 & \beta\frac{e^{\mu x}}{\varepsilon_2(x)} & \frac{\gamma e^{\nu x}}{2\varepsilon_2(x)}\\ 0 & \frac{\gamma e^{\nu x}}{2\varepsilon_2(x)} & \frac{\rho}{2}\end{bmatrix}, \quad (103)$$

and

$$R_1(t) = \int_0^1 \tilde{u}^2(x,t)\frac{e^{-\mu x}}{\varepsilon_1(x)}dx \quad (104)$$

$$R_2(t) = \beta\int_0^1 \tilde{v}^2(x,t)\frac{e^{\mu x}}{\varepsilon_2(x)}dx \quad (105)$$

$$R_3(t) = \gamma\tilde{\eta}(t)\int_0^1 \tilde{v}(x,t)\frac{e^{\nu x}}{\varepsilon_2(x)}dx \quad (106)$$

$$R_4(t) = \frac{\rho}{2}\tilde{\eta}^2(t). \quad (107)$$

Let us introduce the constants

$$\underline{\lambda} = \min_{x\in[0,1]}\lambda_{\min}(P(x)) \quad (108)$$

$$\overline{\lambda} = \max_{x\in[0,1]}\lambda_{\max}(P(x)). \quad (109)$$

Inequality (99) ensures that $P(x)$ is positive definite and symmetric for all $x\in[0,1]$, and hence, using the fact that $\varepsilon_1,\ \varepsilon_2 \in C^2([0,1])$ with $\varepsilon_1(x), \varepsilon_2(x) > 0$, for all $x\in[0,1]$, one can conclude that, $\overline{\lambda},\ \underline{\lambda} > 0$. Therefore,

$$\underline{\lambda}\left(\int_0^1 \left(\tilde{u}^2(x,t)+\tilde{v}^2(x,t)\right)dx+\tilde{\eta}^2(t)\right) \leq V(t) \leq \overline{\lambda}\left(\int_0^1 \left(\tilde{u}^2(x,t)+\tilde{v}^2(x,t)\right)dx+\tilde{\eta}^2(t)\right). \quad (110)$$

Using (104)–(107) we get along the solutions of system (87)–(92) that

$$\begin{aligned}\dot{R}_1(t) &= -2\int_0^1 \tilde{u}(x,t)\tilde{u}_x(x,t)e^{-\mu x}dx + 2\int_0^1 \tilde{u}(x,t)\tilde{v}(x,t)\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}dx\\ &= \left(q^2\tilde{v}^2(0,t)-e^{-\mu}\tilde{u}^2(1,t)\right)-\mu\int_0^1 \tilde{u}^2(x,t)e^{-\mu x}dx\\ &\quad + 2\int_0^1 \tilde{u}(x,t)\tilde{v}(x,t)\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}dx \end{aligned} \quad (111)$$

$$\begin{aligned}\dot{R}_2(t) &= 2\beta\int_0^1 \tilde{v}(x,t)\tilde{v}_x(x,t)e^{\mu x}dx + 2\beta\int_0^1 \tilde{u}(x,t)\tilde{v}(x,t)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x}dx\\ &= \beta\left(k_P^2e^{\mu}\tilde{v}^2(0,t)+2k_Pk_Ie^{\mu}\tilde{v}(0,t)\tilde{\eta}(t)+k_I^2e^{\mu}\tilde{\eta}^2(t)\right.\\ &\quad \left.-\tilde{v}^2(0,t)\right)-\mu\beta\int_0^1 \tilde{v}^2(x,t)e^{\mu x}dx\\ &\quad + 2\beta\int_0^1 \tilde{u}(x,t)\tilde{v}(x,t)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x}dx \end{aligned} \quad (112)$$

$$\begin{aligned}\dot{R}_3(t) &= \gamma\tilde{\eta}(t)\int_0^1 \tilde{v}_x(x,t)e^{\nu x}dx + \gamma\tilde{v}(0,t)\int_0^1 \tilde{v}(x,t)\frac{e^{\nu x}}{\varepsilon_2(x)}dx\\ &\quad + \gamma\tilde{\eta}(t)\int_0^1 \tilde{u}(x,t)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\nu x}dx\\ &\leq \gamma\tilde{\eta}(t)\left(e^{\nu}\left(-k_P\tilde{v}(0,t)-k_I\tilde{\eta}(t)\right)-\tilde{v}(0,t)\right)\\ &\quad -\nu\gamma\tilde{\eta}(t)\int_0^1 \tilde{v}(x,t)e^{\nu x}dx\\ &\quad + \frac{\kappa\gamma}{2}\tilde{v}^2(0,t)+\frac{\gamma}{2\kappa}\int_0^1 \tilde{v}^2(x,t)\frac{e^{2\nu x}}{\varepsilon_2^2(x)}dx\\ &\quad + \gamma\tilde{\eta}(t)\int_0^1 \tilde{u}(x,t)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\nu x}dx \end{aligned} \quad (113)$$

$$\dot{R}_4(t) = \rho\tilde{v}(0,t)\tilde{\eta}(t), \quad (114)$$

where we used integration by parts in the first terms of (111)–(113) and Young's inequality in the second term of (113). Using (102), (111)–(114) we get

$$\begin{aligned}\dot{V}(t) &\leq -\begin{bmatrix}\tilde{v}(0,t)\\ \tilde{\eta}(t)\end{bmatrix}^\top M_1\begin{bmatrix}\tilde{v}(0,t)\\ \tilde{\eta}(t)\end{bmatrix}\\ &\quad -\int_0^1 \begin{bmatrix}\tilde{u}(x,t)\\ \tilde{v}(x,t)\\ \tilde{\eta}(t)\end{bmatrix}^\top M(x)\begin{bmatrix}\tilde{u}(x,t)\\ \tilde{v}(x,t)\\ \tilde{\eta}(t)\end{bmatrix}dx\\ &\quad - e^{-\mu}\tilde{u}^2(1,t)-\theta V(t), \end{aligned} \quad (115)$$

where M_1 is given by (93) and

$$M(x) = \begin{bmatrix}A(x) & B^\top(x)\\ B(x) & C\end{bmatrix}, \quad (116)$$

with

$$A(x) = \begin{bmatrix}A_1(x) & A_2(x)\\ A_3(x) & A_4(x)\end{bmatrix}, \quad (117)$$

where

$$A_1(x) = \left(\mu-\frac{\theta}{\varepsilon_1(x)}\right)e^{-\mu x} \quad (118)$$

$$A_2(x) = -\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}-\beta\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x} \quad (119)$$

$$A_3(x) = -\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}-\beta\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x} \quad (120)$$

$$A_4(x) = \beta\left(\mu-\frac{\theta}{\varepsilon_2(x)}\right)e^{\mu x}-\frac{\gamma}{2\kappa}\frac{e^{2\nu x}}{\varepsilon_2^2(x)} \quad (121)$$

$$B(x) = \begin{bmatrix}-\frac{\gamma}{2}\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\nu x} & \frac{\gamma}{2}\left(\nu-\frac{\theta}{\varepsilon_2(x)}\right)e^{\nu x}\end{bmatrix} \quad (122)$$

$$C = \frac{\gamma-\theta\rho}{2}. \quad (123)$$

Using the Schur complement of C in $M(x)$ and (100), (123) one has that $M(x) \geq 0$ for all $x \in [0, 1]$, if and only if

$$M_2(x) = A(x) - B^\top(x)C^{-1}B(x) \geq 0. \tag{124}$$

Thus, if $M_1 \geq 0$ and $M_2(x) \geq 0$, for all $x \in [0, 1]$, one has

$$\dot{V}(t) \leq -e^{-\mu}\tilde{u}^2(1, t) - \theta V(t), \tag{125}$$

and hence, $V(t) \leq e^{-\theta t}V(0)$, for all $t \geq 0$. Combining this relation with (110) the proof is complete.

Remark 2. A control law with an integral action is designed in [5] for 2×2 hyperbolic systems. Stability of the closed-loop system is proved using a diagonal Lyapunov functional. Here the non-diagonal term in the Lyapunov functional is needed for proving stability using a quadratic Lyapunov functional. Indeed, let us assume that the Lyapunov functional is diagonal. We can write it as

$$V(t) = \int_0^1 \left(q_1(x)\tilde{u}^2(x, t) + q_2(x)\tilde{v}^2(x, t)\right) dx + \frac{\rho}{2}\tilde{\eta}^2(t), \tag{126}$$

where the functions q_1 and q_2 belong to $C^1([0, 1])$ with $q_1(x)$, $q_2(x) > 0$, for all $x \in [0, 1]$. The time derivative of V along the solutions of system (87), (88) with boundary conditions (89)–(92) is given by

$$\begin{aligned}\dot{V}(t) = &\begin{bmatrix}\tilde{v}(0, t)\\ \tilde{\eta}(t)\end{bmatrix}^\top \begin{bmatrix}D_1 & D_2\\ D_3 & D_4\end{bmatrix}\begin{bmatrix}\tilde{v}(0, t)\\ \tilde{\eta}(t)\end{bmatrix}\\ &+ \int_0^1 \begin{bmatrix}\tilde{u}(x, t)\\ \tilde{v}(x, t)\end{bmatrix}^\top E(x)\begin{bmatrix}\tilde{u}(x, t)\\ \tilde{v}(x, t)\end{bmatrix} dx\\ &- q_1(1)\varepsilon_1(1)\tilde{u}^2(1, t),\end{aligned} \tag{127}$$

where

$$D_1 = q_1(0)\varepsilon_1(0)q^2 - q_2(0)\varepsilon_2(0) + q_2(1)\varepsilon_2(1)k_P^2 \tag{128}$$

$$D_2 = \frac{1}{2}\left(q_2(1)\varepsilon_2(1)k_P k_I + \rho\right) \tag{129}$$

$$D_3 = \frac{1}{2}\left(q_2(1)\varepsilon_2(1)k_P k_I + \rho\right) \tag{130}$$

$$D_4 = q_2(1)\varepsilon_2(1)k_I^2 \tag{131}$$

$$E(x) = \begin{bmatrix}(q_1(x)\varepsilon_1(x))_x & q_1(x)\gamma_1(x) + q_2(x)\gamma_2(x)\\ q_1(x)\gamma_1(x) + q_2(x)\gamma_2(x) & -(q_2(x)\varepsilon_2(x))_x\end{bmatrix}. \tag{132}$$

Using (127) and (131) one can conclude that when $k_I \neq 0$ the inequality $\dot{V} \leq 0$ cannot be satisfied for any $\begin{bmatrix}\tilde{u} & \tilde{v} & \tilde{\eta}\end{bmatrix}^\top$.

In [3], it is proved that if there exist two boundary controllers for 2×2 linear hyperbolic systems of the form (87), (88) such that the functional

$$\begin{aligned}V(t) = &\int_0^1 q_1(x)\tilde{u}^2(x, t) + q_2(x)\tilde{v}^2(x, t)\\ &+ q_3(x)\tilde{u}(x, t)\tilde{v}(x, t)dx,\end{aligned} \tag{133}$$

along the solutions of the system (87), (88) with the state $(\tilde{u}, \tilde{v})$ satisfies $\dot{V} < 0$ then the cross term q_3 between $\tilde{u}$ and $\tilde{v}$ is necessarily identically zero. However, in the case of stabilization of 2×2 linear hyperbolic systems of the form (87), (88) with a PI control law that we consider here, the cross term (106) in the Lyapunov functional (102) between the integral state $\tilde{\eta}$ of the controller and the state of the plant $\tilde{v}$ is necessary (as explained above) for proving stability of the overall closed-loop system consisting of the plant state $(\tilde{u}, \tilde{v})$ and the integral state $\tilde{\eta}$, using the Lyapunov functional defined in (102) (although a cross term between $\tilde{u}$ and $\tilde{v}$ is not necessary).

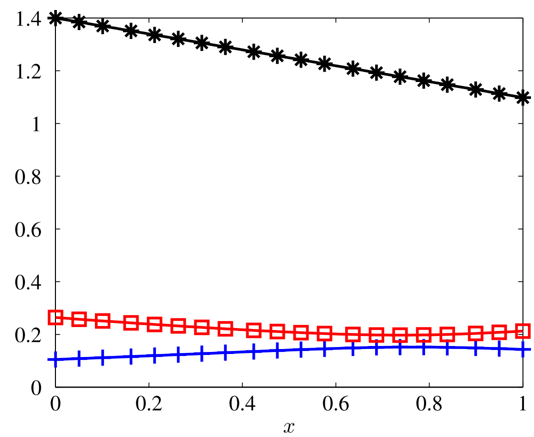

Fig. 1. Evolution of the eigenvalues of (94) as a function of x (square and cross markers), and of the determinant of $P(x)$ in (103) (star marker) for Example 2.

As explained in Remark 2 the non-diagonal term in the Lyapunov functional is crucial for proving stability using a quadratic Lyapunov functional. However, this term adds considerable complexity in verifying analytically that the matrices (93), (94) are positive definite and that (99) holds. In general, the positivity of M_1 and M_2 must be verified numerically. Yet, from the expression of M_1 we see that a necessary condition is that k_I is strictly positive. In addition, from (93) it is evident that k_P must satisfy $|k_P| < 1$. Note that from (93)–(100) it seems possible that the positivity of M_1 and M_2 may depend on the values of γ_1, γ_2, and q. Next, we numerically verify the conditions of Theorem 2 for the system from Example 1.

Example 2 (Example 1 *Continued*)**.** We set in (61)–(63)

$$\varepsilon_1 = 3 \tag{134}$$

$$\varepsilon_2 = 6 \tag{135}$$

$$\tau = 5 \tag{136}$$

$$q = 0.2, \tag{137}$$

and choose U in (90) according to (91) with

$$k_P = 0.1 \tag{138}$$

$$k_I = 1.0583, \tag{139}$$

in order to stabilize the zero equilibrium of (61)–(63). We verify numerically that the conditions of Theorem 2 are satisfied with

$$(\beta, \kappa, \mu, \nu, \theta, \rho, \gamma) = (0.7, 0.2, 0.5, 0.2, 0.7, 2, 2). \tag{140}$$

From (93) we get that

$$M_1 = \begin{bmatrix}0.4485 & 0\\ 0 & 0.2926\end{bmatrix} > 0. \tag{141}$$

The verification of the positive definiteness of matrix (94) is more delicate due to its dependence on x. Fig. 1 shows the evolution of the eigenvalues of $M_2(x)$ and the determinant of matrix (103), which remain positive for all $x \in [0, 1]$.

3.2. Compensation in the output of in-domain and boundary disturbances

Let us assume that there exist some disturbances $d_1, d_2 \in C^1([0, 1])$ on the right-hand side of (11), (12), respectively and some disturbances $d_3, d_4 \in \mathbb{R}$ on the right-hand side of (15), (16), respectively. The error system (87)–(90) becomes

$$\tilde{u}_t + \varepsilon_1(x)\tilde{u}_x = \gamma_1(x)\tilde{v} + d_1(x) \tag{142}$$

$$\tilde{v}_t - \varepsilon_2(x)\tilde{v}_x = \gamma_2(x)\tilde{u} + d_2(x) \tag{143}$$

$$\tilde{u}(0,t) = q\tilde{v}(0,t) + d_3 \tag{144}$$

$$\tilde{v}(1,t) = \tilde{U}(t) + d_4, \tag{145}$$

with $\tilde{U}(t)$ and $\tilde{\eta}(t)$ given by (91) and (92) respectively. The equilibrium of the perturbed system (142)–(144) and (90) is the solution of the following ordinary differential equation

$$Z'(x) = F(x)Z(x) + G(x), \tag{146}$$

where $F(x) = \begin{bmatrix} 0 & \frac{\gamma_1(x)}{\varepsilon_1(x)} \\ -\frac{\gamma_2(x)}{\varepsilon_2(x)} & 0 \end{bmatrix}$ and $G(x) = \begin{bmatrix} \frac{d_1(x)}{\varepsilon_1(x)} \\ -\frac{d_2(x)}{\varepsilon_2(x)} \end{bmatrix}$, with boundary conditions

$$Z_1(0) = d_3 \tag{147}$$

$$Z_2(0) = 0. \tag{148}$$

The ordinary differential equation (146) together with the boundary conditions (147), (148) is a well-posed initial value problem for x. The equilibrium depends on d_1, d_2, and d_3. Let us denote this equilibrium by $\tilde{u}_{ss}(x; d_1, d_2, d_3)$, $\tilde{v}_{ss}(x; d_1, d_2, d_3)$. From (145) it follows that the equilibrium value of $\tilde{U}$, namely $\tilde{U}_{ss}$, satisfies

$$\tilde{U}_{ss} = \tilde{v}_{ss}(1; d_1, d_2, d_3) - d_4. \tag{149}$$

Using (91) and (148) with $Z = \left[\tilde{u}_{ss}, \tilde{v}_{ss}\right]^{\top}$, it follows from (149) that the equilibrium value of $\tilde{\eta}$, namely $\tilde{\eta}_{ss}$, satisfies

$$\tilde{\eta}_{ss} = -\frac{\tilde{v}_{ss}(1; d_1, d_2, d_3) - d_4}{k_I}. \tag{150}$$

Let us define

$$\bar{u}(x,t) = \tilde{u}(x,t) - \tilde{u}_{ss}(x; d_1, d_2, d_3) \tag{151}$$

$$\bar{v}(x,t) = \tilde{v}(x,t) - \tilde{v}_{ss}(x; d_1, d_2, d_3) \tag{152}$$

$$\bar{\eta}(t) = \tilde{\eta}(t) - \tilde{\eta}_{ss}. \tag{153}$$

Using (146) with $Z = \left[\tilde{u}_{ss}, \tilde{v}_{ss}\right]^{\top}$ together with (142), (143) it is shown that the variables $\bar{u}$ and $\bar{v}$ satisfy

$$\bar{u}_t + \varepsilon_1(x)\bar{u}_x = \gamma_1(x)\bar{v} \tag{154}$$

$$\bar{v}_t - \varepsilon_2(x)\bar{v}_x = \gamma_2(x)\bar{u}. \tag{155}$$

Setting $x = 0$ in (151), (152), and using (144), (147), and (148) we get that

$$\bar{u}(0,t) = q\bar{v}(0,t). \tag{156}$$

Setting $x = 1$ in (152) and using (145), (91), and (150) we get $\bar{v}(1,t) = -k_P\tilde{v}(0,t) - k_I\tilde{\eta}(t) + k_I\tilde{\eta}_{ss}$. Using (152) for $x = 0$ together with (148) and (153) we arrive at

$$\bar{v}(1,t) = -k_P\bar{v}(0,t) - k_I\bar{\eta}(t). \tag{157}$$

Using (153) and the fact that

$$\bar{v}(0,t) = \tilde{v}(0,t), \tag{158}$$

relation (92) becomes

$$\dot{\bar{\eta}}(t) = \bar{v}(0,t). \tag{159}$$

Under the assumptions of Theorem 2 the zero equilibrium of (154)–(157) and (159) is exponentially stable.

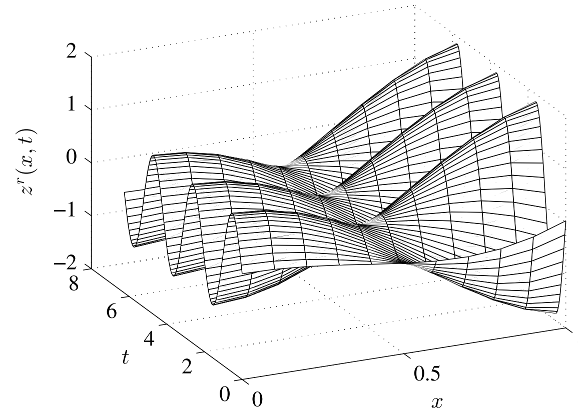

Fig. 2. Solution to the trajectory generation problem for system (70)–(72) with parameters (160)–(163).

4. Simulations

4.1. Trajectory generation for a wave PDE

In this subsection, we illustrate our trajectory generation methodology with a wave PDE of the form (70)–(72). We choose the parameters of the system as

$$\varepsilon = 1 \tag{160}$$

$$h = 1 \tag{161}$$

$$b = -1 \tag{162}$$

$$g = 0. \tag{163}$$

The reference for the output is chosen as $\zeta(t) = \sin(3t)$. As in Example 1, this choice of parameters gives $c_2 = c_4 = 0$, and hence $\gamma_1 = 0$. Therefore, using relations (65)–(68) we obtain

$$L^{\alpha\alpha}(x,\xi) = 0 \tag{164}$$

$$L^{\alpha\beta}(x,\xi) = 0 \tag{165}$$

$$L^{\beta\alpha}(x,\xi) = -\frac{1}{2}\exp\left(\frac{x+\xi}{2}\right) \tag{166}$$

$$L^{\beta\beta}(x,\xi) = -\frac{1}{2}\exp\left(\frac{x-\xi}{2}\right). \tag{167}$$

The reference trajectory z^r for system (70)–(72) is given by

$$z^r(x,t) = \frac{1}{37}\left(19\sin(3t+3x) - 3\exp(x)\cos(3t-3x) + 3\cos(3t+3x) + 18\exp(x)\sin(3t-3x)\right), \tag{168}$$

which gives the following reference input

$$W^r(t) = \frac{57}{37}\left(\cos(3t+3) - \exp(1)\cos(3t-3)\right) + \frac{9}{37}\left(\exp(1)\sin(3t-3) - \sin(3t+3)\right). \tag{169}$$

Fig. 2 shows the evolution of the reference trajectory z^r. Fig. 3 shows the evolution of the spatial derivative of z^r and, in particular, the control effort $W^r(t) = z^r_x(1,t)$ given by (169).

4.2. Trajectory tracking

In this subsection, a simulation study for the system from Examples 1 and 2 is presented. The numerical approximation of the solution is computed with a two-step variant of the Lax–Friedrichs (LxF) method [35]. The reference for the output is chosen as

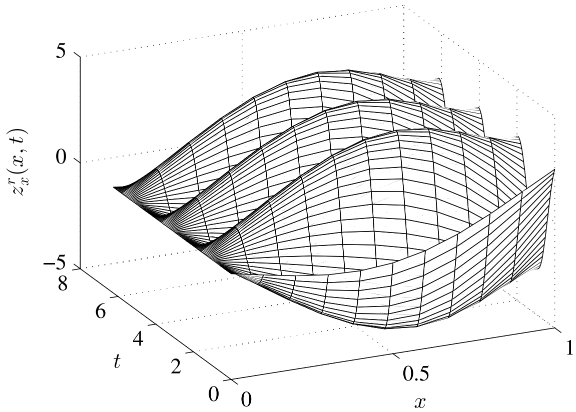

Fig. 3. The spatial derivative of the reference trajectory of Fig. 2. Note in particular the reference input $W^r(t) = z_x^r(1, t)$ given by (169).

$y^r(t) = \cos(t)$. We add disturbances at the right-hand side of (61), (62) given by

$$d_1(x) = 0.5\exp(x) \tag{170}$$
$$d_2(x) = \cos(2x), \tag{171}$$

together with constant additive disturbances on the boundary conditions (63), (64) given by

$$d_3 = 0.5 \tag{172}$$
$$d_4 = 0.5. \tag{173}$$

The initial conditions for u and v are chosen as the reference initial conditions given by (19), (20) for $t = 0$, perturbed by spatially-varying errors as

$$u(x, 0) = u^r(x, 0) + \sin(x) \tag{174}$$
$$v(x, 0) = v^r(x, 0) + \cos(x), \tag{175}$$

and the initial condition for $\tilde{\eta}$ is chosen such that $U(0) = v(1, 0)$, that is,

$$\tilde{\eta}(0) = \frac{U^r(0) - v(1, 0) + k_P(v^r(0, 0) - v(0, 0))}{k_I}. \tag{176}$$

Fig. 4 shows that the output of the system $v(0, t)$ follows the desired trajectory under the PI controller given by

$$\begin{aligned} U(t) &= \cos\left(t + \frac{1}{6}\right) + \frac{3}{229}\left(\exp\left(-\frac{1}{45}\right)\sin(t) \right.\\ &\quad \left. - \exp\left(-\frac{1}{15}\right)\sin\left(t - \frac{1}{3}\right)\right)\\ &\quad + \frac{2}{1145}\left(\exp\left(-\frac{1}{45}\right)\cos(t) - \exp\left(-\frac{1}{15}\right)\cos\left(t - \frac{1}{3}\right)\right)\\ &\quad + \frac{6}{241}\left(\exp\left(\frac{1}{45}\right)\sin\left(t + \frac{1}{6}\right) - \exp\left(-\frac{1}{45}\right)\sin(t)\right)\\ &\quad + \frac{8}{1205}\left(\exp\left(\frac{1}{45}\right)\cos\left(t + \frac{1}{6}\right) - \exp\left(-\frac{1}{45}\right)\cos(t)\right)\\ &\quad - k_P\left(v(0, t) - \cos(t)\right) - k_I\tilde{\eta}(t), \end{aligned} \tag{177}$$

with gains (138), (139), and $\dot{\tilde{\eta}}(t) = v(0, t) - \cos(t)$. One can also observe that with only a P controller (i.e., when $k_I = 0$ in (177)) there is a steady-state tracking error. Fig. 5 shows the evolution of the state v.

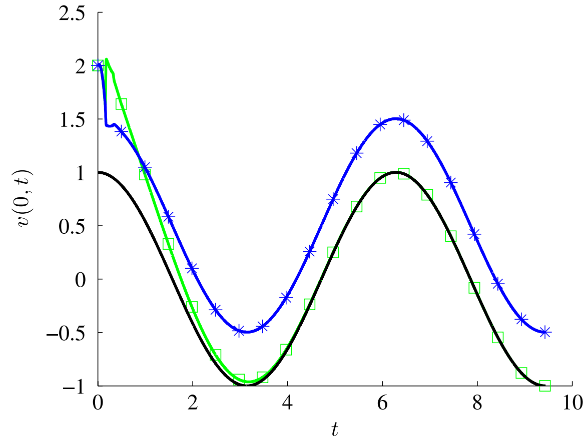

Fig. 4. The output $v(0, t)$ of system (61)–(64) with parameters (134)–(137) under the control law (177) with gains (138), (139) (square marker) and with gains (138), $k_I = 0$ (star marker) for the initial conditions (174)–(176). The single line is the reference output $y^r(t) = \cos(t)$.

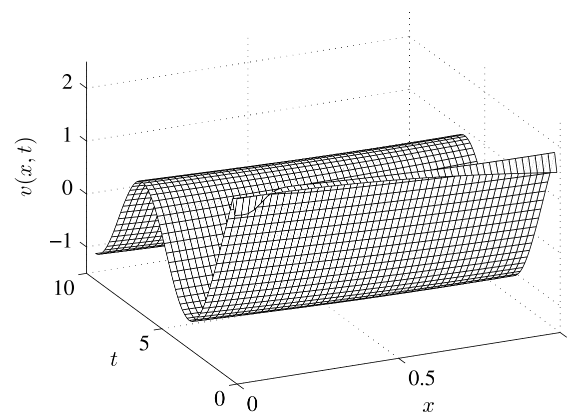

Fig. 5. Evolution of the state v of system (61)–(64) with parameters (134)–(137) under the control law (177) with gains (138), (139) for the initial conditions (174)–(176).

5. Conclusions

We presented solutions to the trajectory generation and tracking problems for general 2×2 systems of first-order linear hyperbolic PDEs. We solved the motion planning problem with backstepping and the trajectory tracking problem with PI control. We proved exponential stability of the closed-loop system by constructing a Lyapunov functional.

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