Input-to-State Stability and Inverse Optimality of Linear Time-Varying-Delay Predictor Feedbacks

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***Abstract*—For linear systems with time-varying input delay and additive disturbances we show that the basic predictor feedback control law is inverse optimal, with respect to a meaningful differential game problem, and establish its robustness to constant multiplicative perturbations appearing at the system input. Both of these properties of the basic predictor feedback controller have not been established so far, even for the constant-delay case. We then show that the basic predictor feedback controller, when applied through a low-pass filter, is again inverse optimal and study its input-to-state stability as well as its robustness, to the low-pass filter time constant properties. All of the stability and inverse optimality proofs are based on the infinite-dimensional backstepping transformation, which allows us to construct appropriate Lyapunov functionals. A numerical example is also provided.**

***Index Terms*—Delay systems, disturbance attenuation, input-to-state stabilization, inverse optimality, linear predictor feedback, robustness.**

I. Introduction

In many control applications one has to deal with systems subject to disturbances [1]–[5]. Time-varying delays also appear in numerous real-world applications, such as, supply networks [6], [7], irrigation channels [8], and robot control [9], among other applications [10]. This triggers efforts toward the development of methodologies for robust stabilization that is especially challenging for systems with input delays. Here, we show that for linear systems with time-varying input delay, the predictor feedback design methodology [10], [11] provides robust, input-to-state stabilizing [2], [3], [12]–[14], and inverse optimal delay-compensating feedback laws.

The input-to-state stability (ISS) concept, introduced by Sontag [2], has played a central role in the development of techniques for robust stabilization of systems with disturbances. Moreover, the inverse optimality approach originated by Kalman and introduced into robust nonlinear control by Freeman [15] based on the control Lyapunov function concept [16]. The inverse optimality concept is of significant practical importance since it allows the design of optimal control laws, which may minimize/maximize a physical quantity of interest and which may possess certain robustness margins, without the need to solve a Hamilton–Jacobi–Isaacs partial differential equation (PDE) (as in the case of the direct optimal control problem) that may not be possible to solve [3]. The problem of robust stabilization becomes more complex in the presence of input delays, even for linear systems. For example, although for ordinary differential equation systems it was shown that the ISS gain (with respect to plant disturbances) can be made arbitrarily small, this is not possible in the presence of input delays [14].

Yet, delay-robustness of the basic constant-delay predictor feedback controller was proved in [17], whereas the problem of inverse optimal redesign for a low-pass filtered version of the basic predictor-based controller was studied in [4]. Moreover, for linear systems with a time-varying input delay, under the basic time-varying-delay predictor feedback controller introduced in [18], a Lyapunov functional was constructed for the closed-loop system and exponential stability was established in [19]. Other robust predictor-based control designs for linear systems with time-varying input delay can be found in [20], [21]. Predictor-based control designs for nonlinear systems with time-varying input delays [9], [22]–[24], as well as wave actuator dynamics with moving boundaries [25]–[28] also exist.

In this paper, robustness of the basic time-varying-delay predictor feedback control law to constant multiplicative perturbations appearing at the system input is established, which may be viewed as a gain margin derivation result. In addition, it is shown that the basic predictor feedback design is inverse optimal, with respect to a meaningful differential game problem. Both of these results are novel even for the constant-delay case. We then consider the case in which the predictor feedback controller is applied through a low-pass filter and prove its robustness, to the first-order input dynamics, as well as its ISS property. Moreover, we show that the low-pass filtered version of the predictor feedback controller is inverse optimal as well. All of our proofs employ the backstepping transformation and the resulting Lyapunov functionals. A numerical example of a second-order, unstable linear system is also presented.

Notation. We use the common definitions of class $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ functions from [11]. For a vector $X \in R^n$, $|X|$ denotes its usual Euclidean norm. For a matrix $A = (a_{ij})_{n\times m}$, $\|A\|$ denotes the induced matrix norm. Given a $\phi(t) : R_+ \to R$, $\phi^{-1}(t)$ denotes the inverse of the function $\phi(t)$; $\phi'(t)$ or $\dot{\phi}(t)$ denotes the derivative of it. For a scalar function $u(\cdot, t) \in L_2(0,1)$, $\|u(t)\|_{L_2}$ denotes the norm given by $(\int_0^1 u^2(x,t)dx)^{1/2}$. For a scalar function $U \in L_2(\phi(t), t)$, $\|U(t)\|_{L_2}$ denotes the norm given by $(\int_{\phi(t)}^t U^2(\theta)d\theta)^{1/2}$. With $\frac{h}{s+h}\{f(t)\}$ we denote the function of time that is the output of the lag transfer function operator $\frac{h}{s+h}\{\cdot\}$ acting on signal $f(t)$.

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II. System Description and Assumption

Consider the linear system given by

$$\dot{X}(t) = AX(t) + B_1 U(t - D(t)) + B_2\delta(t) \quad (1)$$

where $X \in R^n$ is the state, $U \in R$ is the input delayed by $D(t)$ units of time, and $\delta \in R^l$ is a continuous, bounded disturbance signal. Matrices A, B_1, and B_2 are of compatible dimensions and (A, B_1) is a completely controllable pair. Denote

$$\phi(t) = t - D(t). \quad (2)$$

Following [19], we make the following assumption.

Assumption 1: The function $\phi(t)$ given by (2) is continuously differentiable and satisfies

$$\phi(t) < t, \quad \text{for all} \quad t \geq 0 \quad (3)$$

$$\phi'(t) > 0, \quad \text{for all} \quad t \geq 0. \quad (4)$$

Moreover,

$$\pi_0 = \frac{1}{\sup\limits_{\theta \geq \phi^{-1}(0)} (\theta - \phi(\theta))} > 0 \quad (5)$$

$$\pi_1 = \frac{1}{\sup\limits_{\theta \geq \phi^{-1}(0)} \phi'(\theta)} > 0 \quad (6)$$

$$\frac{1}{\pi_2} = \inf_{\theta \geq \phi^{-1}(0)} \phi'(\theta) > 0 \quad (7)$$

$$\frac{1}{\pi_3} = \inf_{\theta \geq \phi^{-1}(0)} (\theta - \phi(\theta)) > 0. \quad (8)$$

Remark 1: Relation (4) guarantees that the delay function satisfies $\dot{D}(t) < 1$, for all $t \geq 0$, i.e., the delay is not increasing at a rate higher than unity. This condition guarantees that there exists a strictly increasing inverse function of ϕ, that is, it guarantees that the prediction time $\phi^{-1}(t)$ exists and is strictly increasing for all $t \geq 0$ (or, in other words, that the control signal never reverses its direction). Since the prediction horizon satisfies $\phi^{-1}(t) - t = D(\phi^{-1}(t))$ the delay is needed to be known in advance. Given that $\phi^{-1}(t) - t \leq \frac{1}{\pi_0}$, for all $t \geq 0$, which follows from (5), one can conclude that the delay is needed to be known at most $\frac{1}{\pi_0}$ s in advance.

III. Gain-Robustness and Inverse Optimality of the Basic Predictor Feedback Controller

We give a basic predictor feedback control law for system (1) as follows:

$$U(t) = \frac{c}{c+1} U_1(t) = U^*(t) \quad (9)$$

where (see [18], [19])

$$U_1(t) = k e^{A(\phi^{-1}(t) - t)} X(t) + k \int_{\phi(t)}^{t} e^{A(\phi^{-1}(t) - \phi^{-1}(\theta))} B_1 \frac{U(\theta)}{\phi'(\phi^{-1}(\theta))} d\theta \quad (10)$$

$c > 0$ is sufficiently large, and the vector k is selected so that $A + B_1 k$ is Hurwitz. We will prove that the closed-loop system (1), (9) is ISS.

We introduce next an equivalent representation of system (1) that is employed in the subsequent stability analysis. Using a transport PDE representation for the actuator state, we rewrite system (1) as follows:

$$\dot{X}(t) = AX(t) + B_1 u(0,t) + B_2 \delta(t) \quad (11)$$

$$u_t(x,t) = \lambda(x,t) u_x(x,t),\ x \in [0,1] \quad (12)$$

$$u(1,t) = U(t) \quad (13)$$

where

$$u(x,t) = U\left(\phi\left(x\left(\phi^{-1}(t) - t\right) + t\right)\right) \quad (14)$$

and

$$\lambda(x,t) = \frac{1 + x\left(\frac{d(\phi^{-1}(t))}{dt} - 1\right)}{\phi^{-1}(t) - t} \quad (15)$$

with $\phi(t)$ given by (2). It is not difficult to find that (10) is written in terms of $u(y,t)$ as follows:

$$U_1(t) = k e^{A(\phi^{-1}(t) - t)} X(t) + k \int_0^1 e^{A(1-y)(\phi^{-1}(t) - t)} B_1 u(y,t) (\phi^{-1}(t) - t) dy. \quad (16)$$

A. ISS Analysis for the Closed-Loop System

Theorem 1: Consider system (11)–(13), together with the control law (9), (16). Under Assumption 1, there exists $c^* > 0$ such that the closed-loop system is ISS for all $c > c^*$, that is, there exist $R > 0, \overline{\lambda} > 0$, and a class $\mathcal{K}$ function $\widetilde{\chi}(r) = \iota r^2$, for some $\iota > 0$, such that for all $c > c^*$

$$\Omega(t) \leq R\Omega(0) e^{-\overline{\lambda} t} + \widetilde{\chi}\left(\sup_{0 \leq \tau \leq t} |\delta(\tau)|\right), \text{ for all } t \geq 0 \quad (17)$$

with

$$\Omega(t) = |X(t)|^2 + \|u(t)\|_{L_2}^2. \quad (18)$$

Proof: The infinite-dimensional backstepping transformation is defined as follows:

$$w(x,t) = u(x,t) - k e^{Ax(\phi^{-1}(t) - t)} X(t) - k \int_0^x e^{A(x-y)(\phi^{-1}(t) - t)} B_1 u(y,t) (\phi^{-1}(t) - t) dy \quad (19)$$

for all $x \in [0,1]$, and the gain vector k is selected so that $A + B_1 k$ is Hurwitz. With (19), we have

$$\begin{aligned} w_t(x,t) &= u_t(x,t) - kAx\left(\frac{d(\phi^{-1}(t))}{dt} - 1\right) e^{Ax(\phi^{-1}(t) - t)} X(t) \\ &\quad - k e^{Ax(\phi^{-1}(t) - t)} \left(AX(t) + B_1 u(0,t) + B_2 \delta(t)\right) \\ &\quad - k \int_0^x e^{A(x-y)(\phi^{-1}(t) - t)} (A(x-y)(\phi^{-1}(t) - t) + I) \\ &\quad \times \left(\frac{d(\phi^{-1}(t))}{dt} - 1\right) B_1 u(y,t) dy \\ &\quad - k \int_0^x e^{A(x-y)(\phi^{-1}(t) - t)} B_1 u_t(y,t) \left(\phi^{-1}(t) - t\right) dy \\ &= u_t(x,t) - k e^{Ax(\phi^{-1}(t) - t)} B_2 \delta(t) \end{aligned}$$

$$
\begin{aligned}
&-\left(1+x\left(\frac{d(\phi^{-1}(t))}{dt}-1\right)\right)k[Ae^{Ax(\phi^{-1}(t)-t)}X(t)\\
&+A\int_0^x e^{A(x-y)(\phi^{-1}(t)-t)}B_1u(y,t)(\phi^{-1}(t)-t)dy\\
&+B_1u(x,t)]
\end{aligned}\tag{20}
$$

where we have used integration by parts, and

$$
\begin{aligned}
&\lambda(x,t)w_x(x,t)\\
&=\lambda(x,t)u_x(x,t)-(\phi^{-1}(t)-t)\lambda(x,t)\\
&\quad\times k[Ae^{Ax(\phi^{-1}(t)-t)}X(t)\\
&\quad+A\int_0^x e^{A(x-y)(\phi^{-1}(t)-t)}B_1u(y,t)(\phi^{-1}(t)-t)dy\\
&\quad+B_1u(x,t)].
\end{aligned}\tag{21}
$$

So under the backstepping transformation (19), with the help of (20), (21), (12), and (15), system (11)–(13) is transformed to the target system as follows:

$$\dot{X}(t)=(A+B_1k)X(t)+B_1w(0,t)+B_2\delta(t)\tag{22}$$

$$w_t(x,t)=\lambda(x,t)w_x(x,t)-ke^{Ax(\phi^{-1}(t)-t)}B_2\delta(t)\tag{23}$$

$$
\begin{aligned}
w(1,t)&=u(1,t)-ke^{A(\phi^{-1}(t)-t)}X(t)\\
&-k\int_0^1 e^{A(1-y)(\phi^{-1}(t)-t)}B_1u(y,t)(\phi^{-1}(t)-t)dy.
\end{aligned}\tag{24}
$$

The inverse backstepping transformation of w is defined as follows[1]:

$$
\begin{aligned}
u(x,t)&=w(x,t)+ke^{(A+B_1k)x(\phi^{-1}(t)-t)}X(t)+k\\
&\times\int_0^x e^{(A+B_1k)(x-y)(\phi^{-1}(t)-t)}B_1w(y,t)(\phi^{-1}(t)-t)dy
\end{aligned}\tag{25}
$$

for all $x\in[0,1]$. Under the inverse transformation (25), the target system (22)–(24) is transformed to system (11)–(13). Using (19) and (25), after some calculations that incorporate the employment of Young's and Cauchy–Schwarz's inequalities, with the help of (5), we get

$$
\begin{aligned}
\frac{|X(t)|^2+\|u(t)\|_{L_2}^2}{\max\{\overline{\beta}_1,1+\overline{\beta}_2\}}&\le|X(t)|^2+\|w(t)\|_{L_2}^2\\
&\le\max\{\overline{\alpha}_1,1+\overline{\alpha}_2\}(|X(t)|^2+\|u(t)\|_{L_2}^2)
\end{aligned}\tag{26}
$$

where

$$
\begin{aligned}
\overline{\alpha}_1&=3\left(1+|k|^2|B_1|^2\frac{e^{2|A|/\pi_0-1}}{2\pi_0|A|}\right)\\
\overline{\alpha}_2&=3|k|^2\pi_0\frac{e^{2|A|/\pi_0-1}}{2|A|}\\
\overline{\beta}_1&=3\left(1+|k|^2|B_1|^2\frac{e^{2|A+B_1k|/\pi_0-1}}{2\pi_0|A+B_1k|}\right)\\
\overline{\beta}_2&=3|k|^2\pi_0\frac{e^{2|A+B_1k|/\pi_0-1}}{2|A+B_1k|}.
\end{aligned}\tag{27}
$$

With the help of (13), (24) and (9), (16), one has

$$w(1,t)=U(t)-U_1(t)=-\frac{1}{c+1}U_1(t).\tag{28}$$

[1]The fact that (25) is the inverse of (19) can be seen in various ways, such as, for example, by direct substitution and using integration by parts as well as changing the order of integration in the double integral.

From (16) and (5), with Young's and Cauchy–Schwarz's inequalities, we get

$$
\begin{aligned}
U_1(t)^2&\le 2|k|^2e^{\frac{2|A|}{\pi_0}}|X(t)|^2+\frac{2|k|^2}{\pi_0^2}e^{\frac{2|A|}{\pi_0}}|B_1|^2\|u(t)\|_{L_2}^2\\
&\le 2|k|^2e^{\frac{2|A|}{\pi_0}}\left(1+\frac{|B_1|^2}{\pi_0^2}\right)(|X(t)|^2+\|u(t)\|_{L_2}^2).
\end{aligned}\tag{29}
$$

With (26) and (29), we have

$$U_1(t)^2\le\overline{a}(|X(t)|^2+\|w(t)\|_{L_2}^2)\tag{30}$$

where

$$\overline{a}=2|k|^2e^{\frac{2|A|}{\pi_0}}\left(1+\frac{|B_1|^2}{\pi_0^2}\right)\max\{\overline{\beta}_1,1+\overline{\beta}_2\}.\tag{31}$$

Using (28) and (30), it is easy to get

$$w^2(1,t)\le\frac{\overline{a}}{(c+1)^2}(|X(t)|^2+\|w(t)\|_{L_2}^2).\tag{32}$$

Consider a Lyapunov functional

$$V(t)=X(t)^TPX(t)+\frac{a_1}{2}\int_0^1 e^{bx}w^2(x,t)dx\tag{33}$$

where P is the solution of the Lyapunov equation

$$P(A+B_1k)+(A+B_1k)^TP=-Q\tag{34}$$

where Q is a positive definite matrix and the constants $a_1>0$ and $b>0$ are determined later. With (33), the derivative of $V(t)$ along the solutions of system (22)–(24) satisfies the following equality:

$$
\begin{aligned}
\dot{V}(t)=&-X^T(t)QX(t)+2X^T(t)PB_1w(0,t)\\
&+2X^T(t)PB_2\delta(t)+\frac{a_1}{2}e^b\lambda(1,t)w^2(1,t)\\
&-\frac{a_1}{2}\lambda(0,t)w^2(0,t)\\
&-\frac{a_1}{2}\int_0^1 e^{bx}\left(b\lambda(x,t)+\lambda_x(x,t)\right)w^2(x,t)dx\\
&-a_1\int_0^1 e^{bx}w(x,t)ke^{Ax(\phi^{-1}(t)-t)}B_2\delta(t)dx
\end{aligned}\tag{35}
$$

where we have used integration by parts. Using similar arguments to the proof in [19][2], choosing

$$b\ge(1-\pi_1)\max\left\{1,\frac{1}{\pi_1}\right\}\tag{36}$$

we get $b\lambda(x,t)+\lambda_x(x,t)\ge\pi_0\Lambda_1$, where

$$\Lambda_1=\min\{b-1+\pi_1,(b+1)\pi_1-1\}>0\tag{37}$$

it holds

$$
\begin{aligned}
\dot{V}(t)\le&-X^T(t)QX(t)+2X^T(t)PB_1w(0,t)\\
&+2X^T(t)PB_2\delta(t)+\frac{a_1}{2}e^b\lambda(1,t)w^2(1,t)\\
&-\frac{a_1}{2}\lambda(0,t)w^2(0,t)-\frac{a_1\pi_0\Lambda_1}{2}\int_0^1 e^{bx}w^2(x,t)dx\\
&-a_1\int_0^1 e^{bx}w(x,t)ke^{Ax(\phi^{-1}(t)-t)}B_2\delta(t)dx.
\end{aligned}\tag{38}
$$

[2]In a nutshell, it is shown in [19] that, since $b\lambda(x,t)+\lambda_x(x,t)$ is a linear function in $x\in[0,1]$, it has a minimum either at $x=0$ or $x=1$ and this minimum is positive when b is chosen according to (36).

Noting from (15) that $\lambda(0,t) = \frac{1}{\phi^{-1}(t)-t} \geq \pi_0$ and $\lambda(1,t) = \frac{\frac{d(\phi^{-1}(t))}{dt}}{\phi^{-1}(t)-t} \leq \pi_2\pi_3$ where we use (5)–(8), with the help of Young's inequality, we have

$$\begin{aligned}
\dot{V}(t) \leq& -X^T(t)QX(t) + \frac{2}{a_1\pi_0}|X^T(t)PB_1|^2 \\
&+ \frac{\lambda_{\min}(Q)|X^T(t)PB_2|^2}{4\lambda_{\max}(PB_2B_2^TP)} + \frac{4\lambda_{\max}(PB_2B_2^TP)|\delta|^2}{\lambda_{\min}(Q)} \\
&+ \frac{a_1}{2}e^b\pi_2\pi_3 w^2(1,t) - \frac{a_1}{4}\pi_0\Lambda_1\int_0^1 e^{bx}w^2(x,t)dx \\
&+ \frac{a_1e^b}{\pi_0\Lambda_1}|k|^2 e^{\frac{2\|A\|}{\pi_0}}\|B_2\|^2|\delta(t)|^2 \\
\leq& -\frac{3\lambda_{\min}(Q)}{4}|X(t)|^2 + \frac{2}{a_1\pi_0}|X^T(t)PB_1|^2 \\
&+ \frac{a_1}{2}e^b\pi_2\pi_3 w^2(1,t) - \frac{a_1}{4}\pi_0\Lambda_1\int_0^1 e^{bx}w^2(x,t)dx \\
&+ \left(\frac{4\lambda_{\max}(PB_2B_2^TP)}{\lambda_{\min}(Q)} + \frac{a_1e^b}{\pi_0\Lambda_1}|k|^2 e^{\frac{2\|A\|}{\pi_0}}\|B_2\|^2\right)|\delta(t)|^2
\end{aligned} \tag{39}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues, respectively, of the corresponding matrices. With the help of (32), we have

$$\begin{aligned}
\dot{V}(t) \leq& -\frac{3\lambda_{\min}(Q)}{4}|X(t)|^2 + \frac{2}{a_1\pi_0}|X^T(t)PB_1|^2 \\
&+ \frac{a_1e^b\pi_2\pi_3\overline{a}}{2(c+1)^2}(|X(t)|^2 + \|w(t)\|_{L_2}^2) \\
&- \frac{a_1}{4}\pi_0\Lambda_1\int_0^1 e^{bx}w^2(x,t)dx \\
&+ \left(\frac{4\lambda_{\max}(PB_2B_2^TP)}{\lambda_{\min}(Q)} + \frac{a_1e^b}{\pi_0\Lambda_1}|k|^2 e^{\frac{2\|A\|}{\pi_0}}\|B_2\|^2\right)|\delta(t)|^2 \\
\leq& -\frac{\lambda_{\min}(Q)}{2}\left(1 - \frac{e^b\pi_2\pi_3a_1\overline{a}}{\lambda_{\min}(Q)(c+1)^2}\right)|X(t)|^2 \\
&- \left(\frac{\lambda_{\min}(Q)}{4}|X(t)|^2 - \frac{2}{a_1\pi_0}|X^T(t)PB_1|^2\right) \\
&- \frac{a_1}{4}\pi_0\Lambda_1\left(1 - \frac{2e^b\pi_2\pi_3\overline{a}}{\pi_0\Lambda_1(c+1)^2}\right)\|w(t)\|_{L_2}^2 \\
&+ \left(\frac{4\lambda_{\max}(PB_2B_2^TP)}{\lambda_{\min}(Q)} + \frac{a_1e^b}{\pi_0\Lambda_1}|k|^2 e^{\frac{2\|A\|}{\pi_0}}\|B_2\|^2\right)|\delta(t)|^2.
\end{aligned} \tag{40}$$

Choosing

$$a_1 = \frac{8\lambda_{\max}(PB_1B_1^TP)}{\pi_0\lambda_{\min}(Q)} \tag{41}$$

and $c > c^*$, where

$$c^* = \frac{\sqrt{e^b\overline{a}\pi_2\pi_3\max\left\{\frac{a_1}{\min(Q)}, \frac{2}{\pi_0\Lambda_1}\right\}}}{\sqrt{1-\overline{\mu}}} \tag{42}$$

for some $0 < \overline{\mu} < 1$, we get

$$\begin{aligned}
\dot{V}(t) \leq& -\overline{\mu}\min\left\{\frac{\lambda_{\min}(Q)}{2}, \frac{a_1\pi_0\Lambda_1}{4}\right\}(|X(t)|^2 + \|w(t)\|_{L_2}^2) \\
&+ \left(\frac{4\lambda_{\max}(PB_2B_2^TP)}{\lambda_{\min}(Q)} + \frac{a_1e^b}{\pi_0\Lambda_1}|k|^2 e^{\frac{2\|A\|}{\pi_0}}\|B_2\|^2\right)|\delta(t)|^2.
\end{aligned} \tag{43}$$

With (33), we have

$$\begin{aligned}
&\min\{\lambda_{\min}(P), \frac{a_1}{2}\}(|X(t)|^2 + \|w(t)\|_{L_2}^2) \\
&\leq V(t) \\
&\leq \max\{\lambda_{\max}(P), \frac{a_1e^b}{2}\}(|X(t)|^2 + \|w(t)\|_{L_2}^2).
\end{aligned} \tag{44}$$

Thus, from (43), (44), it holds

$$\dot{V}(t) \leq -\overline{\lambda}V(t) + \overline{\nu}|\delta(t)|^2 \tag{45}$$

with

$$\overline{\lambda} = \frac{\overline{\mu}\min\{\frac{\lambda_{\min}(Q)}{2}, \frac{a_1\pi_0\Lambda_1}{4}\}}{\max\{\lambda_{\max}(P), \frac{a_1e^b}{2}\}} \tag{46}$$

$$\overline{\nu} = \frac{4\lambda_{\max}(PB_2B_2^TP)}{\lambda_{\min}(Q)} + \frac{a_1e^b}{\pi_0\Lambda_1}|k|^2 e^{\frac{2\|A\|}{\pi_0}}\|B_2\|^2. \tag{47}$$

Using (45) and the comparison principle (see e.g., [11]), we arrive at

$$\begin{aligned}
V(t) &\leq e^{-\overline{\lambda}t}V(0) + \frac{\overline{\nu}}{\overline{\lambda}}\sup_{0\leq\tau\leq t}|\delta(\tau)|^2 \\
&\leq e^{-\overline{\lambda}t}V(0) + \frac{\overline{\nu}}{\overline{\lambda}}\left(\sup_{0\leq\tau\leq t}|\delta(\tau)|\right)^2.
\end{aligned} \tag{48}$$

Thus, system (22)–(24) together with the control law (9), (16) is ISS for all $c > c^*$ with $c^* > 0$ given by (42). By (26), (44), and (48), it can be deduced that

$$|X(t)|^2 + \|u(t)\|^2 \tag{49}$$

$$\begin{aligned}
&\leq e^{-\overline{\lambda}t}\frac{\max\{\overline{\beta}_1, 1+\overline{\beta}_2\}}{\min\{\lambda_{\min}(P), \frac{a_1}{2}\}}\max\left\{\lambda_{\max}(P), \frac{a_1e^b}{2}\right\} \\
&\quad\times\max\{\overline{\alpha}_1, 1+\overline{\alpha}_2\}\left(|X(0)|^2 + \|u(0)\|_{L_2}^2\right) \\
&\quad+\frac{\max\{\overline{\beta}_1, 1+\overline{\beta}_2\}}{\min\{\lambda_{\min}(P), \frac{a_1}{2}\}}\frac{\overline{\nu}}{\overline{\lambda}}\left(\sup_{0\leq\tau\leq t}|\delta(\tau)|\right)^2.
\end{aligned} \tag{50}$$

So we get (17) with $\iota = \frac{\max\{\overline{\beta}_1, 1+\overline{\beta}_2\}}{\min\{\lambda_{\min}(P), \frac{a_1}{2}\}}\frac{\overline{\nu}}{\overline{\lambda}}$, $R = \frac{\max\{\overline{\beta}_1, 1+\overline{\beta}_2\}}{\min\{\lambda_{\min}(P), \frac{a_1}{2}\}}\max\{\lambda_{\max}(P), \frac{a_1e^b}{2}\}\max\{\overline{\alpha}_1, 1+\overline{\alpha}_2\}$ and $\overline{\lambda}$ given by (46).

Theorem 2: Consider the closed-loop system (1), (9), (10). Under Assumption 1, there exists $c^* > 0$ such that the closed-loop system is ISS for all $c > c^*$, that is, there exist $\overline{R} > 0, \overline{\lambda} > 0$, and a class $\mathcal{K}$ function $\widetilde{\chi}_1(r) = \widetilde{\iota}r^2$, for some $\widetilde{\iota} > 0$, such that for all $c > c^*$

$$\widetilde{\Omega}(t) \leq \overline{R}\widetilde{\Omega}(0)e^{-\overline{\lambda}t} + \widetilde{\chi}_1\left(\sup_{0\leq\tau\leq t}|\delta(\tau)|\right), \text{ for all } t \geq 0 \tag{51}$$

with

$$\widetilde{\Omega}(t) = |X(t)|^2 + \|U(t)\|_{L_2}^2. \tag{52}$$

Proof: Noting $u(x,t) = U(\phi(x(\phi^{-1}(t)-t)+t))$ and employing the change of variable $\theta = \phi(x(\phi^{-1}(t)-t)+t)$, it can be deduced that

$$\int_{\phi(t)}^{t} U^2(\theta)d\theta = (\phi^{-1}(t) - t) \times \int_0^1 \phi'(x(\phi^{-1}(t) - t) + t)u^2(x,t)dx \quad (53)$$

$$\int_0^1 u^2(x,0)dx = \frac{1}{\phi^{-1}(0)} \int_{\phi(0)}^{0} U^2(\theta)\frac{d\theta}{\phi'(\phi^{-1}(\theta))}. \quad (54)$$

With (53) and (54), we have

$$\begin{aligned}&|X(t)|^2 + \|U(t)\|_{L_2}^2\\ &= |X(t)|^2 + (\phi^{-1}(t) - t)\int_0^1 \phi'(x(\phi^{-1}(t) - t) + t)u^2(x,t)dx\\ &\leq h_1(|X(t)|^2 + \|u(t)\|_{L_2}^2)\end{aligned} \quad (55)$$

with $h_1 = \max\{1, \frac{1}{\pi_0}\sup_{\tau\geq 0}\phi'(\tau)\}$. Using (17) and (55), we have

$$\begin{aligned}&|X(t)|^2 + \|U(t)\|_{L_2}^2\\ &\leq h_1\left(Re^{-\bar{\lambda}t}\left(|X(0)|^2 + \|u(0)\|_{L_2}^2\right) + \widetilde{\chi}(\sup_{0\leq\tau\leq t}|\delta(\tau)|)\right)\\ &\leq h_1 h_2 Re^{-\bar{\lambda}t}\left(|X(0)|^2 + \|U(0)\|_{L_2}^2\right) + h_1\widetilde{\chi}\left(\sup_{0\leq\tau\leq t}|\delta(\tau)|\right)\end{aligned} \quad (56)$$

with $h_2 = \max\{1, \frac{1}{\phi^{-1}(0)\inf_{0\leq\tau\leq\phi^{-1}(0)}\phi'(\tau)}\}$. Denote $\overline{R} = h_1 h_2 R$ and $\widetilde{\iota} = h_1\iota$, we have (51).

B. Inverse Optimality

Theorem 3: Consider the closed-loop system (1), (9), (10). Under Assumption 1, there exists a $c^{**} \geq c^*$ such that for all $c > c^{**}$, the control law (9), (10) minimizes the cost functional:

$$\begin{aligned}J &= \sup_{\delta\in\Xi}\lim_{t\to\infty}(2\beta V(t)\\ &\quad + \int_0^t \left(L(\tau) + \frac{\beta a_1 e^b\lambda(1,\tau)}{c}U^2(\tau) - 4\beta\overline{\nu}|\delta(\tau)|^2\right)d\tau\Big)\end{aligned} \quad (57)$$

where L is a functional of $(X(t), U(\theta))$, for all $t - D(t) \leq \theta \leq t$, such that

$$L(t) \geq \beta\chi\widetilde{\Omega}(t) \quad (58)$$

for an arbitrary $\beta > 0$ and some $\chi > 0$, $a_1, b, V, \overline{\nu}$, and $\widetilde{\Omega}$ are given by (41), (36), (33), (47), and (52), respectively, and Ξ is the set of linear scalar-valued functions of X.

Proof: Choose

$$\begin{aligned}L(t) = &-\frac{\beta a_1 e^b\lambda(1,t)}{c+1}U_1(t)^2 + 2\beta X^T(t)QX(t)\\ &- 4\beta X^T(t)PB_1 w(0,t) + a_1\beta\lambda(0,t)w^2(0,t)\\ &- 2\beta\frac{1}{\overline{\nu}}X^T(t)PB_2B_2^T PX(t)\\ &+ a_1\beta\int_0^1 e^{bx}(b\lambda(x,t) + \lambda_x(x,t))w^2(x,t)dx\\ &- \frac{\beta a_1\pi_0\Lambda_1}{2}\int_0^1 e^{bx}w^2(x,t)dx\end{aligned} \quad (59)$$

where $\pi_0, \Lambda_1, a_1, b, \overline{\nu}, U_1$, and w are given by (5), (37), (41), (36), (47), (10), and (19), respectively, and β is an arbitrary positive scalar. Noting that $\overline{\nu}$ is given by (47), it is easy to know $\overline{\nu} \geq \frac{4\lambda_{\max}(PB_2B_2^TP)}{\lambda_{\min}(Q)}$. With the help of (30) and $\lambda(1,t) \leq \pi_2\pi_3$, after some calculations that involve Young's inequality and the fact that $b\lambda(x,t) + \lambda_x(x,t) \geq \pi_0\Lambda_1$, for all x and t, we get

$$\begin{aligned}L(t) \geq &-\beta\frac{\overline{a}a_1e^b\pi_2\pi_3}{c+1}(|X(t)|^2 + \|w(t)\|_{L_2}^2)\\ &+ 2\beta\lambda_{\min}(Q)|X(t)|^2 - \frac{4\beta}{a_1\pi_0}|X^T(t)PB_1|^2\\ &- \frac{\beta}{2}\lambda_{\min}(Q)|X(t)|^2 + \frac{\beta a_1\pi_0\Lambda_1}{2}\int_0^1 e^{bx}w^2(x,t)dx\\ \geq &-\beta\frac{\overline{a}a_1e^b\pi_2\pi_3}{c+1}(|X(t)|^2 + \|w(t)\|_{L_2}^2)\\ &+ \beta\lambda_{\min}(Q)|X(t)|^2 + \frac{\beta a_1\pi_0\Lambda_1}{2}\int_0^1 e^{bx}w^2(x,t)dx\\ \geq &\,\beta\left(-\frac{\overline{a}a_1e^b\pi_2\pi_3}{c+1} + \min\{\lambda_{\min}(Q), \frac{a_1\pi_0\Lambda_1}{2}\}\right)\\ &\times(|X(t)|^2 + \|w(t)\|_{L_2}^2).\end{aligned} \quad (60)$$

Choosing $c > c^{**}$ where c^{**} is such that

$$c^{**} \geq \max\left\{\frac{\overline{a}a_1e^b\pi_2\pi_3}{\frac{1}{2}\min\{\lambda_{\min}(Q), \frac{a_1\pi_0\Lambda_1}{2}\}}, \frac{\sqrt{e^b\overline{a}\pi_2\pi_3\max\left\{\frac{a_1}{\min(Q)}, \frac{2}{\pi_0\Lambda_1}\right\}}}{\sqrt{1-\overline{\mu}}}\right\} \quad (61)$$

for some $0 < \overline{\mu} < 1$, by (26), (55), and (61), we get from (60) that

$$\begin{aligned}L(t) &\geq \frac{\beta}{2}\min\{\lambda_{\min}(Q), \frac{a_1\pi_0\Lambda_1}{2}\}(|X(t)|^2 + \|w(t)\|_{L_2}^2)\\ &\geq \frac{\frac{\beta}{2}\min\{\lambda_{\min}(Q), \frac{a_1\pi_0\Lambda_1}{2}\}}{\max\{\overline{\beta}_1, 1+\overline{\beta}_2\}}(|X(t)|^2 + \|u(t)\|_{L_2}^2)\\ &\geq \frac{\frac{\beta}{2}\min\{\lambda_{\min}(Q), \frac{a_1\pi_0\Lambda_1}{2}\}}{h_1\max\{\overline{\beta}_1, 1+\overline{\beta}_2\}}(|X(t)|^2 + \|U(t)\|_{L_2}^2).\end{aligned} \quad (62)$$

Hence, (58) is achieved with $\chi = \frac{\frac{1}{2}\min\{\lambda_{\min}(Q), \frac{a_1\pi_0\Lambda_1}{2}\}}{h_1\max\{\overline{\beta}_1, 1+\overline{\beta}_2\}}$. With (35), we have from (62) that

$$\begin{aligned}&L(t)\\ &= -\frac{\beta ca_1e^b\lambda(1,t)}{(c+1)^2}U_1(t)^2\\ &s + \beta a_1e^b\lambda(1,t)\left(w(1,t)^2 - \frac{U_1(t)^2}{(c+1)^2}\right)\\ &\quad - 2\beta\dot{V}(t) + 4\beta X^T(t)PB_2\delta(t)\\ &\quad - 2\beta a_1\int_0^1 e^{bx}w(x,t)ke^{Ax(\phi^{-1}(t)-t)}B_2\delta(t)dx\\ &\quad - 2\beta\frac{1}{\overline{\nu}}X^T(t)PB_2B_2^TPX(t) - \frac{\beta a_1\pi_0\Lambda_1}{2}\int_0^1 e^{bx}w^2(x,t)dx.\end{aligned} \quad (63)$$

Hence, using (9) and (30), we get

$$
\begin{aligned}
L(t)
&= -\frac{\beta c a_1 e^b \lambda(1,t)}{(c+1)^2} U_1(t)^2 \\
&\quad + \beta a_1 e^b \lambda(1,t)\left((U(t)-U_1(t))^2 - \frac{U_1(t)^2}{(c+1)^2}\right) \\
&\quad - 2\beta \dot{V}(t) + 4\beta X^T(t) P B_2 \delta(t) \\
&\quad - 2\beta a_1 \int_0^1 e^{bx} w(x,t) k e^{Ax(\phi^{-1}(t)-t)} B_2 \delta(t) dx \\
&\quad - 2\beta \frac{1}{\overline{\nu}} X^T(t) P B_2 B_2^T P X(t) - \frac{\beta a_1 \pi_0 \Lambda_1}{2} \int_0^1 e^{bx} w^2(x,t) dx \\
&= \frac{\beta a_1 e^b \lambda(1,t)}{c} (U^*(t))^2 \\
&\quad + \beta a_1 e^b \lambda(1,t)\left((U(t)-U^*(t))^2 - \frac{2U(t)U^*(t)}{c}\right) \\
&\quad - 2\beta \dot{V}(t) + 4\beta X^T(t) P B_2 \delta(t) \\
&\quad - 2\beta a_1 \int_0^1 e^{bx} w(x,t) k e^{Ax(\phi^{-1}(t)-t)} B_2 \delta(t) dx \\
&\quad - 2\beta \frac{1}{\overline{\nu}} X^T(t) P B_2 B_2^T P X(t) - \frac{\beta a_1 \pi_0 \Lambda_1}{2} \int_0^1 e^{bx} w^2(x,t) dx.
\end{aligned} \tag{64}
$$

Denoting

$$
\begin{aligned}
\Pi(\delta(\tau))
&= -2\beta a_1 \int_0^1 e^{bx} w(x,\tau) k e^{Ax(\phi^{-1}(\tau)-\tau)} B_2 \delta(\tau) dx \\
&\quad - \frac{\beta a_1 \pi_0 \Lambda_1}{2} \int_0^1 e^{bx} w^2(x,\tau) dx + 4\beta X^T(t) P B_2 \delta(\tau) \\
&\quad - 2\beta \frac{1}{\overline{\nu}} X^T(t) P B_2 B_2^T P X(t) - 4\beta \overline{\nu} |\delta(\tau)|^2
\end{aligned} \tag{65}
$$

it can be deduced from (64) that

$$
\begin{aligned}
&\int_0^t \left(L(\tau) + \frac{\beta a_1 e^b \lambda(1,\tau)}{c} U^2(\tau) - 4\beta \overline{\nu} |\delta(\tau)|^2\right) d\tau \\
&\quad = -2\beta V(t) + 2\beta V(0) \\
&\qquad + \int_0^t \beta a_1 e^b \lambda(1,\tau)\left(1 + \frac{1}{c}\right)(U(\tau) - U^*(\tau))^2 d\tau \\
&\qquad + \int_0^t \Pi(\delta(\tau)) d\tau.
\end{aligned} \tag{66}
$$

So we get

$$
\begin{aligned}
J &= 2\beta V(0) + \int_0^\infty \beta a_1 e^b \lambda(1,\tau) \\
&\quad \times \left(1 + \frac{1}{c}\right)(U(\tau) - U^*(\tau))^2 d\tau + \sup_{\delta \in \Xi} \int_0^\infty \Pi(\delta(\tau)) d\tau.
\end{aligned} \tag{67}
$$

Using Young's inequality in the first integral in (65), we get

$$
\begin{aligned}
\Pi(\delta(\tau)) &\le 2\beta \frac{a_1 e^b}{\pi_0 \Lambda_1} |k|^2 e^{\frac{2\|A\|}{\pi_0}} \|B_2\|^2 |\delta(\tau)|^2 - 2\beta \overline{\nu} |\delta(\tau)|^2 \\
&\quad - 2\beta\left(\overline{\nu}|\delta(\tau)|^2 - 2X^T P B_2 \delta(\tau) + \frac{X^T P B_2 B_2^T P X}{\overline{\nu}}\right).
\end{aligned} \tag{68}
$$

Noting from (47) that $(\frac{a_1 e^b}{\pi_0 \Lambda_1}) |k|^2 e^{\frac{2\|A\|}{\pi_0}} \|B_2\|^2 \le \overline{\nu}$, we then get

$$
\Pi(\delta(\tau)) \le \; -2\beta \left| \sqrt{\overline{\nu}} \delta(\tau) - \frac{B_2^T P X}{\sqrt{\overline{\nu}}} \right|^2 \tag{69}
$$

and, thus, $|\sqrt{\overline{\nu}}\delta(\tau) - \frac{B_2^T P X}{\sqrt{\overline{\nu}}}|^2 = 0$ if and only if

$$
\delta^*(\tau) = \frac{B_2^T P X}{\overline{\nu}}. \tag{70}
$$

Thus,

$$
\sup_{\delta(\tau) \in \Xi} \int_0^\infty \Pi(\delta(\tau)) d\tau = 0 \tag{71}
$$

and the "worst case" disturbance is given by (70). With (67) and (71), we get

$$
\begin{aligned}
J &= 2\beta V(0) \\
&\quad + \int_0^\infty \beta a_1 e^b \lambda(1,\tau)\left(1 + \frac{1}{c}\right)(U(\tau) - U^*(\tau))^2 d\tau.
\end{aligned} \tag{72}
$$

With (5) and (7), $\lambda(1,\tau) \ge \frac{1}{\pi_0 \pi_1} > 0$, for all $\tau \ge 0$, holds so the minimum of (72) is reached with

$$
U(t) = U^*(t) \tag{73}
$$

such that

$$
J = 2\beta V(0). \tag{74}
$$

Remark 2: For the constant-delay case, that is, when $D(t) = D$, the control law (9), where $U_1(t)$ is now defined as follows:

$$
U_1(t) = k e^{AD} X(t) + k \int_{t-D}^t e^{A(t-\theta)} B_1 U(\theta) d\theta \tag{75}
$$

is inverse optimal for the system

$$
\dot{X}(t) = AX(t) + B_1 U(t-D) + B_2 \delta(t). \tag{76}
$$

IV. Stability and Inverse Optimality of Dynamic Implementation of Predictor Feedback Controller

The control law for system (1) is given now by

$$
\begin{aligned}
U(t) &= \frac{h}{s+h}\Big\{k e^{A(\phi^{-1}(t)-t)} X(t) \\
&\quad + k \int_{\phi(t)}^t e^{A(\phi^{-1}(t)-\phi^{-1}(\theta))} B_1 \frac{U(\theta)}{\phi'(\phi^{-1}(\theta))} d\theta\Big\}
\end{aligned} \tag{77}
$$

where $h > 0$ is sufficiently large and the vector k is selected so that $A + B_1 k$ is Hurwitz. The feedback (77) is a low-pass filtered version of the predictor feedback control law [10].

In the constant-delay case, the implementation of predictor feedback control laws may suffer from implementation problems reported in the literature, see, e.g., [29], [30]. Yet, besides the low-pass filtered version of the basic predictor feedback design proposed in [29], there is a rich literature focusing on the implementation and approximation issues of predictor feedback laws, see, e.g., [31], [32]. Consequently, one could in principle try to extend the existent techniques to systems with time-varying input delays.

The control law (77) is written in terms of $u(x,t)$ as follows:

$$u(1,t) = \frac{h}{s+h}\Big\{ke^{A(\phi^{-1}(t)-t)}X(t) + k\int_0^1 e^{A(1-y)(\phi^{-1}(t)-t)}B_1 u(y,t)(\phi^{-1}(t)-t)dy\Big\}. \tag{78}$$

A. ISS Analysis for the Closed-Loop System

Theorem 4: Consider system (11)–(13), together with the control law (78). Under Assumption 1, the closed-loop system is ISS, that is, there are $\overline{h} > 0, g > 0, \mu > 0$, and a class $\mathcal{K}$ function $\chi(r) = \varsigma r^2$, for some $\varsigma > 0$, such that for all $h > \overline{h}$

$$\Gamma(t) \le g\Gamma(0)e^{-\mu t} + \chi\left(\sup_{0\le\tau\le t}|\delta(\tau)|\right), \text{ for all } t \ge 0 \tag{79}$$

with

$$\Gamma(t) = |X(t)|^2 + \|u(t)\|_{L_2}^2 + u^2(1,t). \tag{80}$$

Proof: Consider a Lyapunov functional

$$V(t) = X(t)^T PX(t) + \frac{a_1}{2}\int_0^1 e^{bx}w^2(x,t)dx + \frac{1}{2}w(1,t)^2 \tag{81}$$

where P is the solution of the Lyapunov (34), and the constants $a_1 > 0$ and $b > 0$ are given by (41) and (36), respectively. The derivative of $V(t)$ along the solutions of system (22)–(24) satisfies the following equality

$$\begin{aligned}\dot{V}(t) &= -X^T(t)QX(t) + 2X^T(t)PB_1w(0,t) + 2X^T(t)PB_2\delta(t)\\ &+ \frac{a_1}{2}e^b\lambda(1,t)w^2(1,t) - \frac{a_1}{2}\lambda(0,t)w^2(0,t)\\ &- \frac{a_1}{2}\int_0^1 e^{bx}\left(b\lambda(x,t) + \lambda_x(x,t)\right)w^2(x,t)dx\\ &- a_1\int_0^1 e^{bx}w(x,t)ke^{Ax\left(\phi^{-1}(t)-t\right)}B_2\delta(t)dx\\ &+ w(1,t)w_t(1,t).\end{aligned} \tag{82}$$

From (78) and (24), it follows that

$$u_t(1,t) = -hw(1,t) \tag{83}$$

where $h > 0$ is sufficiently large. After some calculations, defining

$$\begin{aligned}\alpha_1 &= 3 + 3|k|^2\|B_1\|^2\frac{e^{\frac{2\|A\|}{\pi_0}} - 1}{2\pi_0\|A\|} + 3\frac{|k|^2}{\pi_0^2}e^{\frac{2\|A\|}{\pi_0}}\|B_1\|^2\\ \alpha_2 &= 3\left(|k|^2\pi_0\frac{e^{\frac{2\|A\|}{\pi_0}} - 1}{2\|A\|} + |k|^2e^{\frac{2\|A\|}{\pi_0}}\right)\\ \beta_1 &= 3 + 3|k|^2\|B_1\|^2\frac{e^{\frac{2\|(A+B_1k)\|}{\pi_0}} - 1}{2\pi_0\|A+B_1k\|} + 3\frac{|k|^2}{\pi_0^2}e^{\frac{2\|A+B_1k\|}{\pi_0}}\|B_1\|^2\\ \beta_2 &= 3\left(|k|^2\pi_0\frac{e^{\frac{2\|A+B_1k\|}{\pi_0}} - 1}{2\|A+B_1k\|} + |k|^2e^{\frac{2\|A+B_1k\|}{\pi_0}}\right)\end{aligned} \tag{84}$$

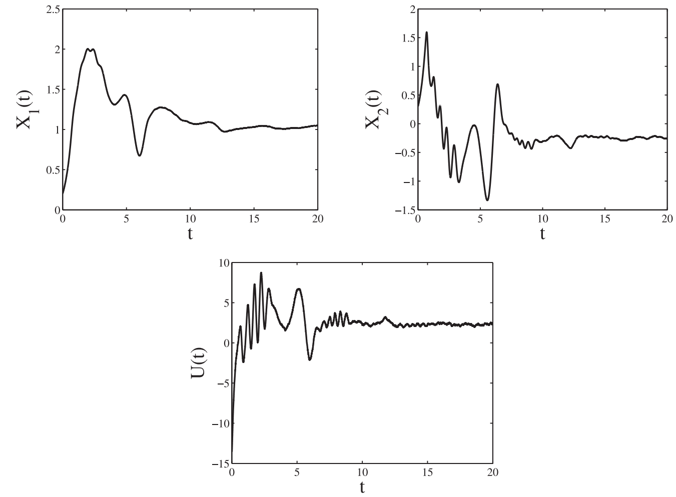

Fig. 1. Responses of the states $X_1(t), X_2(t)$ of system (90) with the control law (91) for initial conditions as $X_1(0) = 0.2, X_2(0) = 0.3$, and $U(\theta) = 0$, for $\theta \in [\phi(0), 0]$ and disturbance $\delta(t) = \text{rand}(1)$.

we have

$$\begin{aligned}&|X(t)|^2 + \|u(t)\|_{L_2}^2 + u^2(1,t)\\ &\le e^{-\mu t}\varrho_1\varrho_2\max\{\beta_2+1,\beta_1,3\}\max\{\alpha_2+1,\alpha_1,3\}\\ &\times\left(|X(0)^2 + \|u(0)\|_{L_2}^2 + u^2(1,0)\right)\\ &+\varrho_1\frac{\nu}{\mu}\max\{\beta_2+1,\beta_1,3\}\left(\sup_{0\le\tau\le t}|\delta(\tau)|\right)^2.\end{aligned} \tag{85}$$

So system (11)–(13), together with the control law (78) is ISS, that is, there exist $g = \varrho_1\varrho_2\max\{\beta_2+1,\beta_1,3\}\max\{\alpha_2+1,\alpha_1,3\} > 0$ and $\varsigma = \varrho_1\frac{\nu}{\mu}\max\{\beta_2+1,\beta_1,3\}$ such that (79) holds.

Theorem 5: Consider the closed-loop system (1), (77). Under Assumption 1, there are $\overline{h} > 0, \overline{g} > 0, \mu > 0$, and a class $\mathcal{K}$ function $\overline{\chi}(r) = \widetilde{\varsigma}r^2$, for some $\widetilde{\varsigma} > 0$, such that for all $h > \overline{h}$

$$\overline{\Gamma}(t) \le \overline{g}\Gamma(0)e^{-\mu t} + \overline{\chi}\left(\sup_{0\le\tau\le t}|\delta(\tau)|\right), \text{ for all } t \ge 0 \tag{86}$$

with

$$\overline{\Gamma}(t) = |X(t)|^2 + \|U(t)\|_{L_2}^2 + U^2(t). \tag{87}$$

Proof: Since the space is limited, the proof is omitted.

B. Inverse Optimality

Theorem 6: Consider the closed-loop system (1) and (77). Under Assumption 1, for any $h > 2\overline{h} + 1$, the feedback (77) minimizes the cost functional:

$$J = \sup_{\delta\in\Xi}\lim_{t\to\infty}\left(2hV(t) + \int_0^t\left(\widetilde{L}(\tau) + \dot{U}(\tau)^2 - 4h\nu|\delta(\tau)|^2\right)d\tau\right) \tag{88}$$

where $V, \overline{h}$, and ν are given by (82), $\widetilde{L}$ is a functional of $(X(t), U(\theta))$, $\theta \in [\phi(t), t]$, such that

$$\widetilde{L}(t) \ge q\Gamma(t) \tag{89}$$

for some $q(h) > 0$ with a property that $q(h) \to \infty$ as $h \to \infty$, and Ξ is the set of linear scalar-valued functions of X.

Proof: Since the space is limited, the proof is omitted.

V. Example

Consider a linear system with time-varying input delay and disturbance as follows:

$$\dot{X}_1(t) = X_2(t) + 0.5\delta(t)$$
$$\dot{X}_2(t) = -2X_1(t) + 3X_2(t) + U(t - D(t) - \triangle D(t)) + \delta(t) \tag{90}$$

where the input delay has a mismatch of $\triangle D(t)$, which can be either positive or negative, and satisfying $D(t) + \triangle D(t) \geq 0$. Choose $k = [-4, -8]$ and assume that $D(t) = \frac{1+t}{1+2t}$ and $\triangle D(t) = 0.5\sin t$. We get $\phi(t) = t - \frac{1+t}{1+2t}$, $\phi^{-1}(t) = \frac{t+\sqrt{t^2+2t+2}}{2}$, $\frac{d\phi^{-1}(t)}{dt} = \frac{\sqrt{t^2+2t+2}+t+1}{2\sqrt{t^2+2t+2}}$. By Theorem 1, the control law for system (90) is given by

$$U(t) = \frac{c[-4, -8]}{c+1}\left(e^{A(\phi^{-1}(t)-t)}X(t) + \int_{\phi(t)}^{t} \frac{d\phi^{-1}(\theta)}{d\theta} e^{A(\phi^{-1}(t)-\phi^{-1}(\theta))} BU(\theta) d\theta\right) \tag{91}$$

where $c > 0$ is sufficiently large. Responses of the states of system (90) under the control law (91) are shown for $c = 100$ in Fig. 1. Disturbance $\delta(t)$ in Fig. 1 is comprised of randomly generated numbers from a uniform distribution in $[0, 1]$. One can observe that the closed-loop system is ISS. By Theorem 3, the control law (91) is inverse optimal.

VI. Conclusion

We have shown that the basic predictor feedback design for linear systems with time-varying input delay and additive disturbances is inverse optimal as well as robust to constant multiplicative perturbations affecting the input. We have also proved that the predictor feedback controller is robust to first-order input dynamics. Moreover, we have established that a low-pass filtered version of the basic predictor feedback controller is inverse optimal and achieves ISS as well. The robustness properties of the basic predictor feedback controller are illustrated by an example.

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