A fast semi-analytic algorithm for computing solutions associated with multiple fixed or mobile capacity restrictions

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Outline

- Motivations
- The LWR model
- Lax-Hopf algorithm
- Fast Lax-Hopf algorithm
- Solutions to joint ODE/PDE problems
- Application to bus control
Motivations

Fast and reliable transit systems can reduce traffic externalities:

- Congestion
- Accidents
- Air pollution

Problems: bus bunching
Motivations

• Traffic is well described by the Lighthill-Whitham-Richards (LWR) model

• Solutions to the LWR model are well known, given a set of initial and boundary conditions

• However: in numerous situations, moving or fixed capacity restrictions affect traffic. These capacity restrictions can furthermore be affected by the surrounding traffic

• How can we efficiently compute the solution to the LWR model with an arbitrary number of moving boundary conditions, following their own dynamics?
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The LWR model

- First derived by Lighthill-Whitham (1955), and extended by Richards (1956)
- First order scalar hyperbolic conservation law

\[
\frac{\partial \rho(t, x)}{\partial t} + \frac{\partial \psi(\rho(t, x))}{\partial x} = 0
\]

- Based on the conservation of vehicles, and on the existence of a relationship between flow and density: \(q = \psi(\rho)\)

\(\psi(.)\) is assumed to be concave

[Newell 93], [Daganzo 03,06]

Source: PeMS
Integral formulation

Equivalently, we can define $M(t,x)$ such that:

$$M(t_2, x_2) - M(t_1, x_1) = \int_{x_1}^{x_2} -\rho(t_1, x)\, dx + \int_{t_1}^{t_2} \psi(\rho(t, x_2))\, dt$$

The function $M(t,x)$ is called *Moskowitz function*. Its spatial derivative is the opposite of the density function; its temporal derivative is the flow function.

$M(t,x)$ satisfies the following Hamilton-Jacobi PDE:

$$\frac{\partial M(t, x)}{\partial t} - \psi \left( -\frac{\partial M(t, x)}{\partial x} \right) = 0$$

[Newell 93], [Daganzo 03,06]
Physical interpretation

M(t,x) can be interpreted as a vehicle label at location x and time t (assuming that no vehicles pass each other).

M(t,x) is also known as the cumulative vehicle number in the traffic flow community.

[Newell 93], [Daganzo 03,06]
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Solution methods

- Existing computational methods:
  - For LWR:
    - Godunov scheme (or equivalently CTM)
    - Wave-front tracking
    - Other finite difference schemes (ENO, WENO)
  - For HJ:
    - Lax Friedrichs schemes (or other numerical schemes)
    - Variational method (dynamic programming)
    - Semi-analytic method (for homogeneous problems), which can be used for both HJ and LWR
Solution method

Based on the classical Lax-Hopf formula

For a boundary data function \( c(\cdot, \cdot) \), the solution \( M_c(\cdot, \cdot) \) is given by:

\[
M_c(t, x) = \inf_{(u, T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+} \left( c(t - T, x + Tu) + T\varphi^*(u) \right)
\]

where \( \varphi^*(u) := \sup_{p \in \text{Dom}(\psi)} [p \cdot u + \psi(p)] \) is the convex transform of \( \psi \)

[Lax 1973] [Aubin Bayen Saint Pierre SIAM SICON 2009] [Claudel Bayen IEEE TAC part II 2010]
Solution method

- **Inf-morphism property**

Let us assume that the boundary data \( c \) is the minimum of a finite number of lower semicontinuous functions:

\[
\forall (t, x) \in [0, t_{\text{max}}] \times [\xi, \chi], c(t, x) = \min_{j \in J} c_{j}(t, x)
\]

The solution associated with the above boundary data function can be decomposed as:

\[
\forall (t, x) \in [0, t_{\text{max}}] \times [\xi, \chi], M_{C}(t, x) = \min_{j \in J} M_{C_{j}}(t, x)
\]
Solution method

- **Inf-morphism property**

Let us assume that the boundary data $c$ is the minimum of a finite number of lower semicontinuous functions:

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$$\forall (t, x) \in [0, t_{\text{max}}] \times [\xi, \chi], \ M_c(t, x) = \min_{j \in J} M_{c_j}(t, x)$$

[Claudel Bayen IEEE TAC part II 2010] [Mazare Dehwah Claudel Bayen, TR-B 2012]
Solutions to affine value conditions

Affine initial condition:

\[ \mathcal{M}_{0,i}(0, x) = \begin{cases} a_i x + b_i & \text{if } x \in [\alpha_i, \alpha_{i+1}] \\ +\infty & \text{otherwise} \end{cases} \]

Affine upstream boundary condition

\[ \gamma_j(t, \xi) = \begin{cases} c_j t + d_j & \text{if } t \in [\gamma_j, \gamma_{j+1}] \\ +\infty & \text{otherwise} \end{cases} \]

Affine downstream boundary condition

\[ \beta_k(t, x) = \begin{cases} e_k t + f_k & \text{if } t \in [\beta_k, \beta_{k+1}] \\ +\infty & \text{otherwise} \end{cases} \]

[Claudel Bayen TAC 2010]
Solutions to affine value conditions

Example of solution: affine initial condition

\[ M_{0,i}(0,x) = \begin{cases} a_i x + b_i & \text{if } x \in [\alpha_i, \alpha_{i+1}] \\ +\infty & \text{otherwise} \end{cases} \]

Lax-Hopf formula:

\[ M_{M_{0,i}}(t,x) = \inf_{u \in \text{Dom}(\varphi^*) \cap [\frac{\alpha_i - x}{t}, \frac{\alpha_{i+1} - x}{t}]} (a_i(x + tu) + b_i + tu \varphi^*(u)), \quad \forall (t,x) \in \mathbb{R}_+^* \times X \]

\[ \forall u \in \text{Dom}(\varphi^*), \quad \zeta_{a_i,b_i,t,x}(u) := a_i(x + tu) + b_i + tu \varphi^*(u) \]

1-D convex optimization

[Claudel Bayen TAC 2010]
Solutions to affine value conditions

Explicit solution to the convex program requires the use of subgradients, since $\varphi^*$ is not necessarily differentiable.

Thus, posing $\mu_0(a_i)$ as any element of $\partial_+ \psi(-a_i)$, we have the following explicit solution:

$$\forall u \in \text{Dom}(\varphi^*),$$
$$\partial_- \zeta_{a_i,b_i,t,x}(u) = \{ w \mid \exists v \in \partial_- \varphi^*(u), \ w = a_i t + vt \}$$
$$:= t \cdot (\{ a_i \} + \partial_- \varphi^*(u))$$

$$M_{0,i}(t,x) = \begin{cases} 
(i) & t \psi(-a_i) + a_i x + b_i \\
& \text{if } u_0(a_i) \in [\frac{a_i}{t} - \frac{\bar{a}_i - x}{t}, \frac{\bar{a}_{i+1} - x}{t}]
(ii) & a_i \bar{a}_i + b_i + t \varphi^*(\frac{\bar{a}_i - x}{t}) \\
& \text{if } u_0(a_i) \leq \frac{a_i}{t} - \frac{\bar{a}_i - x}{t}
(iii) & a_i \bar{a}_{i+1} + b_i + t \varphi^*(\frac{\bar{a}_{i+1} - x}{t}) \\
& \text{if } u_0(a_i) \geq \frac{a_i}{t} - \frac{\bar{a}_i - x}{t}
\end{cases}$$

[Claudel Bayen TAC 2010]
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The Fast Lax-Hopf algorithm

Original Lax-Hopf algorithm

\[ M(t, x) = \min \left( \min_{i, j, k} M_{c_{ini}}^i(t, x), M_{cup}^j(t, x), M_{c_{down}}^k(t, x) \right) \]

**Problem:** number of required operations to compute the solution at a fixed time horizon \( T \) grows linearly with the time horizon

**Solution:** some rules can be derived to eliminate some operations a priori:

**Example:** Let set of \( n_{ini} \) initial conditions be defined. Let us further assume that \( M_{c_{ini}}^i(t', x_{n_{ini}}) \leq M_{c_{ini}}^j(t', x_{n_{ini}}) \) for a time \( t' \geq \frac{x_{n_{ini}} - x_{i+1}}{v_f} \), with \( i < j \). Then:

\[ \forall \, t \geq t', \, M_{c_{ini}}^i(t, x_{n_{ini}}) \leq M_{c_{ini}}^j(t, x_{n_{ini}}) \]

[Simoni Claudel, 2018 (submitted)]
The Fast Lax-Hopf algorithm

Example:

Let set of $n_{ini}$ initial conditions be defined. Let us further assume that $M_{c_{ini}}^i (t', x_{n_{ini}}) \leq M_{c_{ini}}^j (t', x_{n_{ini}})$ for a time $t' \geq \frac{x_{n_{ini}i} - x_{i+1}}{v_f}$, with $i < j$. Then:

$\forall \ t \geq t', M_{c_{ini}}^i (t, x_{n_{ini}}) \leq M_{c_{ini}}^j (t, x_{n_{ini}})$

Proof: subderivatives and inequalities in the sense of intervals

Let $v_i \in \partial_+ \Psi(-a_i)$

$\partial_- M_{c_{ini}}^i (\emptyset, x_{n_{ini}}) = \{ \varphi^* \frac{x_{n_{ini}} - x_{i+1}}{t} - \frac{x_{n_{ini}} - x_i}{t} \}$

$\partial_+ M_{c_{ini}}^i (\emptyset, x_{n_{ini}}) = \{ \varphi^* \frac{x_{n_{ini}} - x_{i+1}}{t} \}$

$\partial_- \varphi^* \frac{x_{n_{ini}} - x_i}{t} \leq \frac{x_{n_{ini}i} - x_{i+1}}{v_i} \leq \frac{x_{n_{ini}i} - x_i}{v_i}$

$\partial_+ \varphi^* \frac{x_{n_{ini}} - x_i}{t} \leq \frac{x_{n_{ini}i} - x_{i+1}}{v_f} \leq \frac{x_{n_{ini}i} - x_i}{v_f}$

[Simoni Claudel, 2018 (submitted)]
Overview

- The classical Lax-Hopf algorithm considers all initial, upstream and downstream blocks to compute the solution at $t + \Delta t$.

\[
N_{c_{ini}}(x_{n_{ini}}, t + \Delta t) \\
n_{min} \leq j \leq n_{ini} \\
\text{where} \\
n_{min} = \min_{x_{i+1} - y_{i} - (t+\Delta t) \geq x_{n_{ini}}} i
\]

\[
N(x_{n_{ini}}, t + \Delta t) \\
d(t) = \frac{M(x_{n_{ini}}, t + \Delta t) - M(x_{n_{ini}}, t)}{\Delta t}
\]

Definition of initial conditions $c_{ini}^{i}$

\[1 \leq i \leq n_{ini}\]

Definition of previous upstream boundary conditions blocks:

$c_{up}^{j}$

Definition of previous downstream boundary conditions blocks:

$c_{down}^{j}$

Computation of $N_{c_{ini}}(x_{n_{ini}}, t + \Delta t)$

Computation of $N_{c_{up}}(x_{n_{ini}}, t + \Delta t)$

for \(k \in [1, \frac{t + \Delta t - x_{n_{ini}}}{v} \Delta t]\)

Computation of $N_{c_{down}}(x_{n_{ini}}, t + \Delta t)$

Determination of actual downstream flow from $d(t)$ using junction model

Define downstream condition block for time interval $[t, t + \Delta t]$
Overview

- The fast Lax-Hopf algorithm drops some initial, upstream and downstream blocks to reduce the number of calculations, though exactness is maintained.
The Fast Lax-Hopf algorithm

The resulting algorithm is much faster than the classical Lax-Hopf algorithm and than the Godunov discretization of the LWR model (CTM model) [Simoni Claudel, 2018 (submitted)]
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A fixed capacity restriction is an internal condition defining a maximum passing flow along a specified fixed location (accidents, traffic lights).

A moving capacity restriction similarly defines a maximum passing flow along a moving boundary (slow vehicle), which usually has a maximum velocity that is lower than the free flow speed.

Both fixed and moving capacity restrictions can be modeled as internal conditions.

**Problem**: internal conditions are not known in advance
Hybrid ODE model for capacity restriction

- The moving capacity restriction is associated with three different modes:
  - A first mode (light traffic) in which traffic is not affected by the moving capacity restriction, since the remaining capacity is sufficient for all flow to pass the restriction unaffected
  - A second mode (medium traffic) in which the restriction affects traffic flow, creating a queue
  - A third mode (slow traffic) in which traffic is slower than the maximum velocity associated with the restriction and thus unaffected
Problem formulation

We want to solve a coupled ODE-PDE system, in which traffic is modeled by the LWR PDE:

\[
\frac{\partial M(x, t)}{\partial t} - \psi \left( -\frac{\partial M(x, t)}{\partial x} \right) = 0
\]

and a finite number of vehicles or fixed obstructions \( i \in I \) each modeled by two states:

\[
\dot{x}_i = s_i \left( -\frac{\partial M(x, t)}{\partial x} \right) \quad \dot{M}_i = p_i \left( -\frac{\partial M(x, t)}{\partial x} \right)
\]

where

\[
s_i(k) = \begin{cases} 
  v_{\text{max},i} & : \frac{\psi(k)}{k} \geq v_{\text{max},i} \\
  \frac{-w \cdot (k-k_i)}{k} & : \text{else}
\end{cases}
\]

\[
p_i(k) = \begin{cases} 
  \min(\psi(k) - k v_{\text{max},i}, r) & : \frac{\psi(k)}{k} \geq v_{\text{max},i} \\
  0 & : \text{else}
\end{cases}
\]
Solution method

- Define a given time step $\Delta t$ associated with the problem
- Define $p$ initial positions (space and time) associated with moving boundary conditions, associated with a maximum velocity $v_k$ and a maximum passing rate $r_k$
- For a given trajectory $k$, with vehicle located at $(x_k, t_k)$, compute $M(x_k, t_k)$ and $M(x_k + v_k\Delta t, t_k + \Delta t)$ (using the Lax-Hopf formula)
- Three cases:
  - If $M(x_k + v_k\Delta t, t_k + \Delta t) < M(x_k, t_k)$ then the moving capacity restriction does not affect traffic (slow traffic)
  - If $M(x_k + v_k\Delta t, t_k + \Delta t) + r_k\Delta t > M(x_k + v_k\Delta t, t_k + \Delta t)$ then the moving capacity restriction does not affect traffic (light traffic)
  - If $M(x_k + v_k\Delta t, t_k + \Delta t) + r_k\Delta t < M(x_k + v_k\Delta t, t_k + \Delta t)$ then the moving capacity restriction affects traffic
Solution method

- The trajectories of the moving capacity restrictions are computed simultaneously using the rules shown earlier.

- Whenever a moving capacity restricts the traffic, we define a corresponding internal condition associated with the velocity and the passing flow of the moving capacity.

[Simoni Claudel, TR-B 2017]
Solution method

- Each internal condition block can only influence the solution in its domain of definition.
- As in some hybrid systems, Zeno effects can occur, when trajectories of moving capacity restrictions are intersecting.
Solution method

• Example of solution
Example forward simulations

5 moving capacity restrictions + 2 traffic lights

8 moving capacity restrictions + 2

trajectories are computed in a few milliseconds (Matlab, laptop)
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Applications to optimization of bus trajectories

- Bus model with passenger arrival rate/boarding model

Number of Passengers at stop n:

\[ N_p(n, t) = \int_0^t \lambda_n dt \]

Alighting:

\[ L_b(m, n) = \mu \cdot N_b(m) \]

Boarding:

\[ B(m, n) = \min\{C(m) - N_b(m) + L_b(m), N_p(n)\} \]

Dwelling Time:

\[ D(m, n) = \max\{B(m, n), L_b(m)\} \times T_b \]
Influence of upstream demand

(a) Demand: 0.4veh/s

(b) Demand: 0.8veh/s
Speed or holding control of buses

In our optimization problem, the objective functions depend on the speed variables $\varphi_j = (v)$, where $v$ correspond to the maximal speed of the bus $j$. We consider three main factors of the bus route system: (i) the total outflow $\langle F \rangle$; (ii) the waiting time of passengers $\langle W \rangle$; (iii) the standard deviation of bus headway $\sigma$.

$$\text{Obj} = -\alpha \langle F \rangle + \beta \langle W \rangle + \gamma \sigma$$
Speed or holding control of buses

- Optimization based on simulated annealing (due to nonlinearities)

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Conclusion

• Fast algorithm for solving the LWR PDE

• Hybrid ODE/PDE model for forward simulation of problems involving multiple moving or fixed capacity restrictions

• Solution method is particularly useful if:
  - only a small number of capacity restrictions are to be modeled (example: bus optimization)
  - or if the objective to be optimized depends only on a low number of solution points (example: travel time optimization)