Control of Nonlinear Delay Systems

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Tutorial Session

CDC 2012

Applications of (Nonconstant) Delay (Nonlinear) Systems

- Control over networks
- Traffic control
- Cooling systems
- Teleoperation
- Milling Processes
- Population Dynamics
- Irrigation channels
- Supply networks
- Rolling mills
- Chemical process control

Nonlinear Control of (Nonconstant) Delay Systems

$$\dot{X}(t) = f\left(X\left(t - D_1(t, X(t))\right), U\left(t - D_2(t, X(t))\right)\right)$$

- $\{D_1(t, X(t)) =$ **constant**$, D_2(t, X(t)) = 0\}$: Jankovic, Karafyllis, Mazenc, Pepe ...
- $\{D_1(t, X(t)) =$ **time-varying or state-dependent**$, D_2(t, X(t)) = 0\}$: ✓
- $\{D_1(t, X(t)) = 0, D_2(t, X(t)) = \mathbf{constant}\}$: ✓**(for large delay)**
- $\{D_1(t, X(t)) = 0, D_2(t, X(t)) =$ **time-varying or state-dependent**$\}$: ✓
- $\{D_1(t, X(t)), D_2(t, X(t)) =$ **time-varying or state-dependent**$\}$: ✓

$U(t - \boldsymbol{D})$

	D=**const.**	$\boldsymbol{D(t)}$	$\boldsymbol{D(X)}$	$\boldsymbol{D(U)}$	**uncertain** $\boldsymbol{D(t, X)}$
Linear	**1**				
Nonlinear					

$U(t - \boldsymbol{D})$

	D=**const.**	$\boldsymbol{D(t)}$	$\boldsymbol{D(X)}$	$\boldsymbol{D(U)}$	**uncertain** $\boldsymbol{D(t, X)}$
Linear	1				
Nonlinear	**2**				

$U(t - \boldsymbol{D})$

	D=**const.**	$\boldsymbol{D(t)}$	$\boldsymbol{D(X)}$	$\boldsymbol{D(U)}$	**uncertain** $\boldsymbol{D(t, X)}$
Linear	1	**3**			
Nonlinear	2				

$U(t - \boldsymbol{D})$

	D=**const.**	$\boldsymbol{D(t)}$	$\boldsymbol{D(X)}$	$\boldsymbol{D(U)}$	**uncertain** $\boldsymbol{D(t, X)}$
Linear	1	3			
Nonlinear	2	**4**			

$U(t - \boldsymbol{D})$

	D=**const.**	$\boldsymbol{D(t)}$	$\boldsymbol{D(X)}$	$\boldsymbol{D(U)}$	**uncertain** $\boldsymbol{D(t,X)}$
Linear	1	3	✓		
Nonlinear	2	4	**5**		

$X(t - \boldsymbol{D})$

	D=**const.**	$\boldsymbol{D(t)}$	$\boldsymbol{D(X)}$	$\boldsymbol{D(U)}$	**uncertain** $\boldsymbol{D(t, X)}$
Linear	✓	✓			
Nonlinear	✓	**6**			

$X(t - \boldsymbol{D})$

	D=**const.**	$\boldsymbol{D(t)}$	$\boldsymbol{D(X)}$	$\boldsymbol{D(U)}$	**uncertain** $\boldsymbol{D(t, X)}$
Linear	✓	✓	✓		
Nonlinear	✓	6	**7**		

$U(t-\boldsymbol{D})$

	D=**const.**	$\boldsymbol{D(t)}$	$\boldsymbol{D(X)}$	$\boldsymbol{D(U)}$	**uncertain** $\boldsymbol{D(t,X)}$
Linear	1	3	✓		**8**
Nonlinear	2	4	5		

$U(t - \boldsymbol{D})$

	D=**const.**	$\boldsymbol{D(t)}$	$\boldsymbol{D(X)}$	$\boldsymbol{D(U)}$	**uncertain** $\boldsymbol{D(t, X)}$
Linear	1	3	✓		8
Nonlinear	2	4	5		**9**

Linear Predictor Feedback for Constant Delay

Linear Systems with Constant Delays (Design)

$$\dot{X}(t) = AX(t) + BU(t - D)$$

Delay-free control law:

$$U(t) = KX(t)$$

Predictor feedback law:

$$\begin{aligned} U(t) &= KP(t) \\ P(t) &= X(t + D) \end{aligned}$$

Challenge: Compute the future state, i.e., $X(t + D)$.

Explicit derivation for linear systems with the variation of constants formula

$$P(t) = X(t) + \int_{t-D}^{t} e^{A(t-\theta)} BU(\theta) d\theta$$

Some History

- Predictor Feedback **Design** for Linear Systems with Constant Delays on the Input and the State : Manitius & Olbrot, Artstein

  Yet, No Analysis was Provided

Linear Systems with Constant Delays (Analysis)

Backstepping Transformation:

$$W(\theta) = U(\theta) - KP(\theta), \quad t - D \le \theta \le t$$

Traget system:

$$\begin{aligned} \dot{X}(t) &= (A + BK)X(t) + BW(t - D) \\ W(t) &= 0, \quad \text{for all } t \ge 0 \end{aligned}$$

Lyapunov-Krasovskii functional:

$$V(t) = X(t)^T P X(t) + b \int_{t-D}^{t} e^{\theta + D - t} W(\theta)^2 d\theta,$$

Theorem 1: $\exists \ \lambda, \ \rho$ such that for all $t \ge 0$

$$|X(t)| + \sqrt{\int_{t-D}^{t} U(\theta)^2 d\theta} \le \rho \left(|X(0)| + \sqrt{\int_{-D}^{0} U(\theta)^2 d\theta} \right) e^{-\lambda t}$$

Nonlinear Predictor Feedback for Constant Delay

Nonlinear Systems with Constant Delays (Design)

$$\dot{X}(t) = f\left(X(t), U(t-D)\right)$$

Delay-free control law:

$$U(t) = \kappa\left(X(t)\right)$$

Predictor feedback law:

$$U(t) \quad = \quad \kappa\left(P(t)\right)$$

Implicit formula for predictor:

$$P(t) = \int_{t-D}^{t} f\left(P(\theta), U(\theta)\right) d\theta$$

Assumptions (Delay-Free Plant)

$\dot{X} = f(X, \omega)$ is forward complete.

$\dot{X} = f(X, \kappa(X) + \omega)$ is ISS, and hence, $\exists$ a dissipative Lyapunov function S

Global asymptotic stability suffices, but for a Lyapunov construction (of the overall infinite-dimensional system) ISS is required

Nonlinear Systems with Constant Delays (Analysis)

Lemma 1 (infinite-dimensional backstepping transformation of the actuator state)

$$\boxed{W(\theta) = U(\theta) - \kappa\left(P(\theta)\right),} \qquad t - D \leq \theta \leq t,$$

transforms the closed-loop system into the "target system"

$$\begin{aligned} \dot{X}(t) &= f\left(X(t), \kappa\left(X(t)\right) + W\left(t - D\right)\right) \\ W(t) &= 0, \quad \forall t \geq 0. \end{aligned}$$

Lemma 2 (g.u.a.s. of target system)
$\exists\ \beta_2 \in \mathcal{KL}$ *s.t.,*

$$|X(t)| + \sup_{t-D\leq\theta\leq\ t} |W(\theta)| \quad \leq \quad \beta_2\left(|X(0)| + \sup_{-D\leq\theta\leq\ 0} |W(\theta)|\,, t\right).$$

Proof: Using the Lyapunov-Krasovskii functional:

$$\begin{aligned} V(t) &= S(X(t)) + b\int_0^{L(t)} \frac{\alpha(r)}{r} dr, \\ L(t) &= \sup_{t-D\leq\theta\leq t} \left|e^{\theta+D-t} W(\theta)\right| \end{aligned}$$

Nonlinear Systems with Constant Delays (Analysis)

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Nonlinear Systems with Constant Delays (Analysis)

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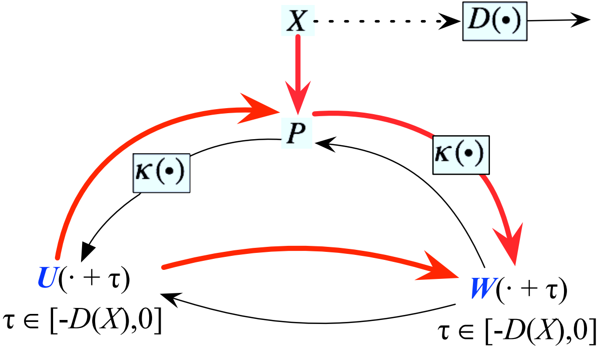
Nonlinear Systems with Constant Delays (Analysis)

Lemma 3 (norm equivalence between the original system and target system)
$\exists \rho_2$, $\alpha_9 \in \mathcal{K}_\infty$ *s.t.*,

$$
\begin{aligned}
|X(t)| + \sup_{t-D\le\theta\le t} |U(\theta)| &\le \alpha_9^{-1}\left(|X(t)| + \sup_{t-D\le\theta\le t} |W(\theta)|\right) \\
|X(t)| + \sup_{t-D\le\theta\le t} |W(\theta)| &\le \rho_2\left(|X(t)| + \sup_{t-D\le\theta\le t} |U(\theta)|\right)
\end{aligned}
$$



X
D(•)
P
κ(•)
κ(•)
U(· + τ)
τ ∈ [-D(X),0]
W(· + τ)
τ ∈ [-D(X),0]

Linear Predictor Feedback for Time-Varying Delay

Linear Systems with Time-Varying Delays (Design)

$$\begin{aligned}\dot{X}(t) &= AX(t) + BU(t - D(t)) \\ \phi(t) &= t - D(t)\end{aligned}$$

Predictor feedback:

$$U(t) = K \underbrace{\left(e^{A\left(\phi^{-1}(t) - t\right)} X(t) + \int_{t-D(t)}^{t} e^{A\left(\phi^{-1}(t) - \phi^{-1}(\theta)\right)} B \frac{U(\theta)}{\phi'\left(\phi^{-1}(\theta)\right)} d\theta \right)}_{P(t) = X\left(\phi^{-1}(t)\right)}$$

Comparison with the constant delay case:

$$U(t) = K \left(e^{AD} X(t) + \int_{t-D}^{t} e^{A(t-\theta)} BU(\theta) d\theta \right)$$

In the time-varying case $D \neq D$

$$\begin{aligned}D &= \phi^{-1}(t) - t \\ D &= t - \phi(t)\end{aligned}$$

Some History

- Predictor Feedback **Design** for Linear Systems with Time-Varying Input Delay : Nihtila

  What about a Lyapunov Functional?

Linear Systems with Time-Varying Delays (Analysis)

Backstepping Transformation:

$$W(\theta) = U(\theta) - KP(\theta), \quad t - D(t) \leq \theta \leq t$$

Traget system:

$$\begin{aligned} \dot{X}(t) &= (A + BK)X(t) + BW(t - D(t)) \\ W(t) &= 0, \quad \text{for all } t \geq 0 \end{aligned}$$

Lyapunov-Krasovskii functional:

$$V(t) = X(t)^T P X(t) + a \int_{t-D(t)}^{t} e^{b \frac{\phi^{-1}(\theta) - t}{\phi^{-1}(t) - t}} W(\theta)^2 d\theta$$

Theorem 3: $\exists\ \lambda$, ρ such that for all $t \geq 0$

$$|X(t)| + \sqrt{\int_{t-D(t)}^{t} U(\theta)^2 d\theta} \leq \rho \left(|X(0)| + \sqrt{\int_{-D(0)}^{0} U(\theta)^2 d\theta} \right) e^{-\lambda t}$$

Nonlinear Predictor Feedback for Time-Varying Delay

Nonlinear Systems with Time-Varying Delays (Design)

$$\begin{aligned} \dot{X}(t) &= f\left(X(t), U(t - D(t))\right) \\ \phi(t) &= t - D(t) \end{aligned}$$

Predictor feedback law:

$$\begin{aligned} U(t) &= \kappa\left(P(t)\right) \\ &= \kappa\left(X\left(\phi^{-1}(t)\right)\right) \end{aligned}$$

Predictor formula:

$$P(t) = X(t) + \int_{\phi(t)}^{t} f\left(P(\theta), U(\theta)\right) \frac{d\theta}{\phi'\left(\phi^{-1}(\theta)\right)}$$

Assumptions (Delay-Free Plant)

$\dot{X} = f(X, \omega)$ is forward complete.

$\dot{X} = f(X, \kappa(X) + \omega)$ is ISS.

Assumptions (Delay)

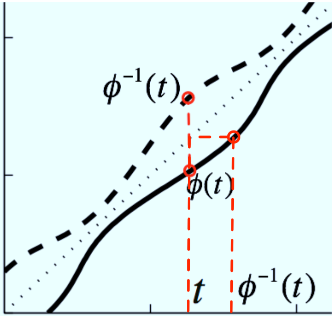
$D(t) \geq 0$ (guarantees the causality of the system)

$D(t) < \infty$ (guarantees that all inputs applied to the plant eventually reach the plant)

$\dot{D}(t) < 1$ (guarantees that the plant never feels input values that are older than the ones it has already felt–input signal direction never reversed)

$\dot{D}(t) > -\infty$ (guarantees that the delay cannot disappear instantaneously, but only gradually)

Achilles heel: $\phi^{-1}(t) > t > \phi(t)$



$D(t)$ needs to be known sufficiently far in advance

$\Rightarrow$ method appears not to be usable for state-dependent delays

Nonlinear Systems with Time-Varying Delays (Analysis)

Backstepping Transformation:

$$W(\theta) = U(\theta) - \kappa(P(\theta)), \quad t - D(t) \leq \theta \leq t$$

Traget system:

$$\begin{aligned} \dot{X}(t) &= f\left(X(t), \kappa(X(t)) + W(t - D(t))\right) \\ W(t) &= 0, \quad \text{for all } t \geq 0 \end{aligned}$$

Lyapunov-Krasovskii functional:

$$\begin{aligned} V(t) &= S(X(t)) + b \int_0^{L(t)} \frac{\alpha(r)}{r} dr, \\ L(t) &= \sup_{t-D(t)\leq\theta\leq t} \left| e^{\frac{\phi^{-1}(\theta)-t}{\phi^{-1}(t)-t}} W(\theta) \right| \end{aligned}$$

Theorem 4: $\exists\ \beta \in \mathcal{KL}$ such that for all $t \geq 0$

$$|X(t)| + \sup_{t-D(t)\leq\theta\leq t} |U(\theta)| \leq \beta\left(|X(0)| + \sup_{-D(0)\leq\theta\leq 0} |U(\theta)|, t\right)$$

Example

$$\begin{aligned}\dot{X}_1(t) &= X_2(t) - X_2(t)^2 U(t - D(t)) \\ \dot{X}_2(t) &= U(t - D(t)) \\ D(t) &= \frac{1+t}{1+2t}\end{aligned}$$

Delay-free control law

$$U(t) = -X_1(t) - 2X_2(t) - \frac{1}{3}X_2(t)^3$$

Predictor feedback

$$U(t) = -P_1(t) - 2P_2(t) - \frac{1}{3}P_2(t)^3$$

$$\begin{aligned}P_1(t) &= \int_{t-D(t)}^{t} \left(P_2(\theta) - P_2(\theta)^2 U(\theta)\right) \frac{d\theta}{\phi'\left(\phi^{-1}(\theta)\right)} + X_1(t) \\ P_2(t) &= \int_{t-D(t)}^{t} U(\theta) \frac{d\theta}{\phi'\left(\phi^{-1}(\theta)\right)} + X_2(t)\end{aligned}$$

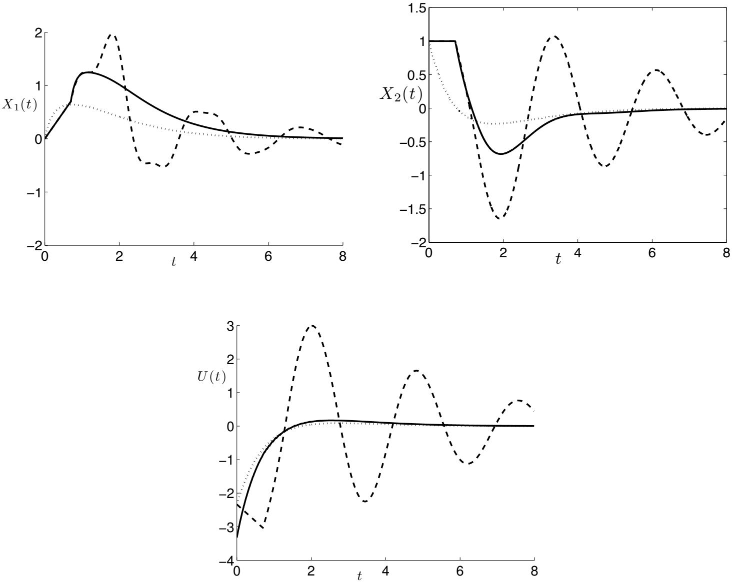


Figure: Controller "kicks in" at $t = \phi^{-1}(0) = \frac{1}{\sqrt{2}}$

State-Dependent Delay

Non-holonomic Unicycle with Distance-Dependent Delay (1)

Nonholonomic Unicycle

$$
\begin{aligned}
\dot{x}(t) &= v\left(t - D\left(x(t), y(t)\right)\right)\cos\left(\theta(t)\right)\\
\dot{y}(t) &= v\left(t - D\left(x(t), y(t)\right)\right)\sin\left(\theta(t)\right)\\
\dot{\theta}(t) &= \omega\left(t - D\left(x(t), y(t)\right)\right)
\end{aligned}
$$

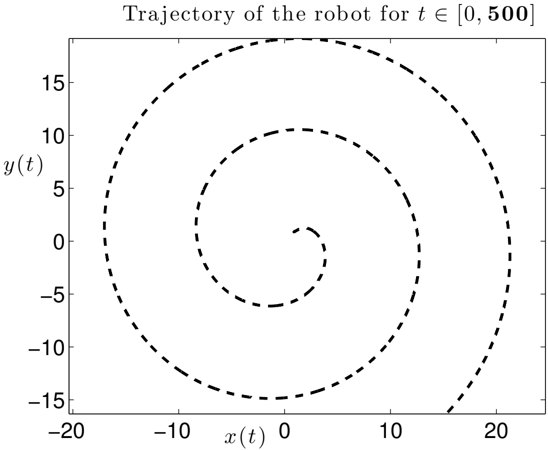
Delay that grows with the distance relative to the reference position

$$D\left(x(t), y(t)\right) = x(t)^2 + y(t)^2$$

A time-varying controller due to Pomet (1992) is

$$
\begin{aligned}
\omega(t) &= -5P(t)^2\cos\left(3t\right) - P(t)Q(t)\left(1 + 25\cos\left(3t\right)^2\right) - \theta(t)\\
v(t) &= -P(t) + 5Q(t)\left(\sin\left(3t\right) - \cos\left(3t\right)\right) + Q(t)\omega(t)\\
P(t) &= x(t)\cos\left(\theta(t)\right) + y(t)\sin\left(\theta(t)\right)\\
Q(t) &= x(t)\sin\left(\theta(t)\right) - y(t)\cos\left(\theta(t)\right)
\end{aligned}
$$

Non-holonomic Unicycle with Distance-Dependent Delay (2)



The trajectory of the robot with the uncompensated controller with initial conditions $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $-x(0)^2 - y(0)^2 \leq s \leq 0$.

Nonlinear Systems with State-Dependent Delay

$$\dot{X}(t) = f\left(X(t), U\left(t - D\left(X(t)\right)\right)\right)$$

Main challenge:

$$P(t) = X\left(t + D\left(P(t)\right)\right)$$

Predictor Feedback

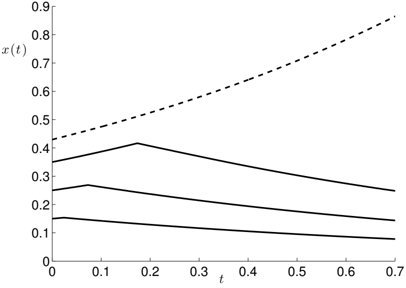
$$\begin{aligned} P(t) &= X(t) + \int_{t-D(X(t))}^{t} \frac{f\left(P(s), U(s)\right) ds}{1 - \nabla D\left(P(s)\right) f\left(P(s), U(s)\right)} \\ &= X\left(\sigma(t)\right) \\ \overbrace{\sigma(t)}^{\phi^{-1}(t)} &= t + D\left(P(t)\right) \end{aligned}$$

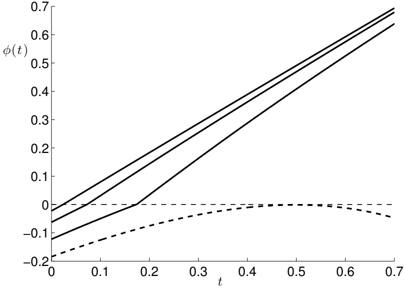
$$U(t) = \kappa\left(P(t)\right)$$

Global Stabilization is not Possible in General

$$\dot{X}(t) = X(t) + U\overbrace{\left(t - X(t)^2\right)}^{\phi(t)}$$

with $U(\theta) = 0$, for all $-X(0)^2 \leq \theta \leq 0$.
The control signal never kicks in for $X(0) \geq X^* = \frac{1}{\sqrt{2e}} = 0.43$





The state of the system with the delay-compensated controller and four different initial conditions $X(0) = 0.15, 0.25, 0.35, X^*$.

Why the Results are not Global

The feasibility condition that the *delay rate is less than one* must hold to ensure that the control signal reaches the plant and that the control remains bounded

The solutions of the system and the initial conditions must satisfy

$$\mathcal{F}_c : \qquad \nabla D\left(P(\theta)\right) f\left(P(\theta), U(\theta)\right) < c, \quad \text{for all } \theta \geq -D\left(X(0)\right)$$

for $c \in (0, 1]$. We refer to $\mathcal{F}_1$ as the *feasibility condition* of the controller.

Assumptions (Delay-Free Plant)

$\dot{X} = f(X, \omega)$ is forward complete

$\dot{X}(t) = f(X(t), \kappa(X(t)) + \omega(t))$ is ISS

Assumptions (Delay)

$D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$

The rest of the assumptions are satisfied by restricting the initial conditions

Theorem 5:

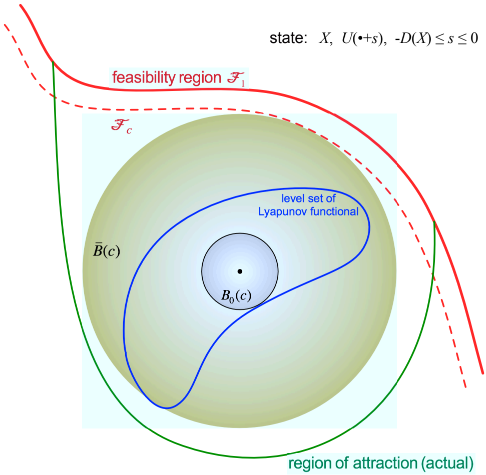
$\exists \psi \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that for all initial conditions that satisfy

$$B_0(c): \qquad |X(0)| + \sup_{-D(X(0)) \leq \theta \leq 0} |U(\theta)| < \psi(c)$$

for some $0 < c < 1$,

$$|X(t)| + \sup_{t-D(X(t)) \leq \theta \leq t} |U(\theta)| \leq \beta\left(|X(0)| + \sup_{-D(X(0)) \leq \theta \leq 0} |U(\theta)|, t\right)$$

The Infinite-Dimensional State-Space



Sets arising in the proof of the Theorem in the infinite-dimensional state space $\mathbb{R}^n \times C[t - D(X(t)), t)$. $B_0(c)$: the ball of initial conditions allowed in the proof of the theorem. $\bar{B}(c)$: the ball inside which the ensuing solutions are trapped.

(X, U)

$\Downarrow$

(X, W)

$\Downarrow$

(X, U)

Lemmas 1–3 (from the constant delay case apply here as well)

Lemma 4 (finding a ball $\bar{B}$ around the origin and within the feasibility region)
$\exists \bar{\rho}_c \in \mathcal{KC}_\infty$ *s.t.* $\mathcal{F}_c$ $(0 < c < 1)$ *is satisfied by all solutions that satisfy*

$$\boxed{\bar{B}(c): \qquad |X(t)| + \sup_{t-D(X(t))\leq\theta\leq t} |U(\theta)| < \bar{\rho}_c(c, c)} \qquad \forall t \geq 0.$$

Lemma 5 (ball B_0 of initial conditions s.t. all solutions are confined in $\bar{B} \subset \mathcal{F}_c$)
$\exists \psi_{\mathrm{RoA}} \in \mathcal{K}$ *s.t. for all initial conditions in* $B_0(c)$*, the solutions remain in* $\bar{B}(c) \subset \mathcal{F}_c$ *for some* $0 < c < 1$.

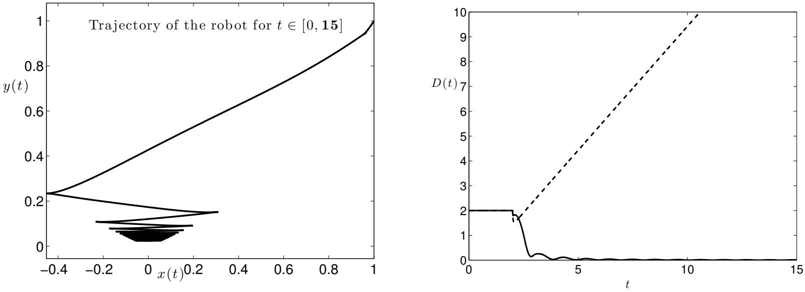
Non-holonomic Unicycle Revisited (1)

$$\begin{aligned}
\omega(t) &= -5P(t)^2\cos(3\sigma(t)) - P(t)Q(t)\left(1+25\cos(3\sigma(t))^2\right) - \Theta(t) \\
v(t) &= -P(t) + 5Q(t)\left(\sin(3\sigma(t)) - \cos(3\sigma(t))\right) + Q(t)\omega(t) \\
P(t) &= X(t)\cos(\Theta(t)) + Y(t)\sin(\Theta(t)) \\
Q(t) &= X(t)\sin(\Theta(t)) - Y(t)\cos(\Theta(t))
\end{aligned}$$

With the predictors

$$\begin{aligned}
X(t) &= x(t) + \int_{t-x(t)^2-y(t)^2}^{t} g(s)v(s)\cos(\Theta(s))\,ds \\
Y(t) &= y(t) + \int_{t-x(t)^2-y(t)^2}^{t} g(s)v(s)\sin(\Theta(s))\,ds \\
\Theta(t) &= \theta(t) + \int_{t-x(t)^2-y(t)^2}^{t} g(s)\omega(s)ds \\
\sigma(t) &= t + X(t)^2 + Y(t)^2 \\
g(s) &= \frac{1}{1-2\left(X(s)v(s)\cos(\Theta(s)) + Y(s)v(s)\sin(\Theta(s))\right)}
\end{aligned}$$

Non-holonomic Unicycle Revisited (2)



The trajectory of the robot, with the compensated controller and the delay function with the compensated controller (solid line) and the uncompensated controller (dashed line) with initial conditions $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $-x(0)^2 - y(0)^2 \leq s \leq 0$.

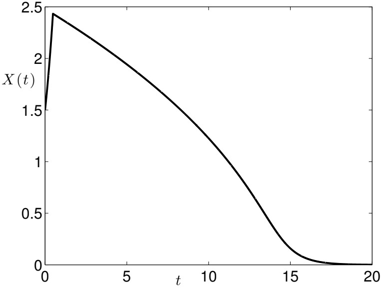
When a Global Result is Possible

$$\nabla D(X)\, f(X, \omega) < c < 1$$

is satisfied, for all $(X, \omega) \in \mathbb{R}^{n+1}$.

When a Global Result is Possible (An Example)

$$\dot{X}(t) = \frac{X(t) + U\left(t - D\left(X(t)\right)\right)}{U\left(t - D\left(X(t)\right)\right)^2 + 1}, \quad \text{with} \quad D\left(X(t)\right) = \frac{1}{4}\log\left(X(t)^2 + 1\right)$$



The state of the system with the delay-compensated controller and initial conditions $X(0) = 1.5$, $U(\theta) = 0$, for all $-\frac{1}{4}\log\left(X(0)^2 + 1\right) \leq \theta \leq 0$. After the controller kicks in, $X(t)$ decays according to $\dot{X}(t) = -\frac{X(t)}{1+4X(t)^2}$.

Forward Completeness and ISS are NOT Necessary!

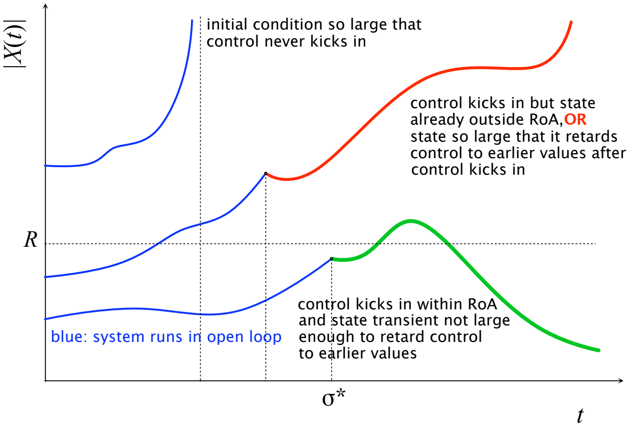
Assumption (Delay-Free Plant)

$\dot{X} = f(X, \omega)$ is locally stabilizable, i.e., $\exists \kappa$, $R > 0$ and $\beta^* \in \mathcal{KL}$ such that, the system $\dot{X} = f(X, \kappa(t, X))$ satisfies

$$|X(t)| \leq \beta^* (|X(0)|, t), \quad t \geq 0,$$

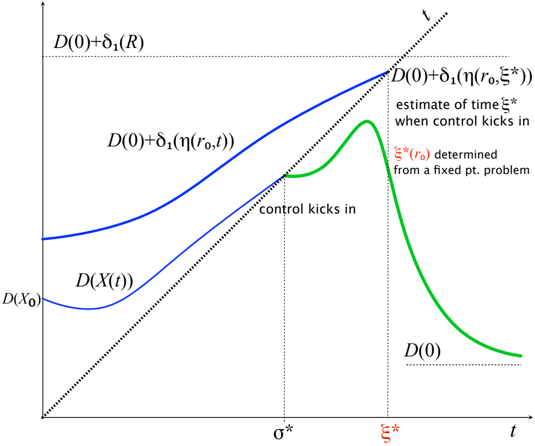
for all $|X(0)| \leq R$.

Why the Results are not Global



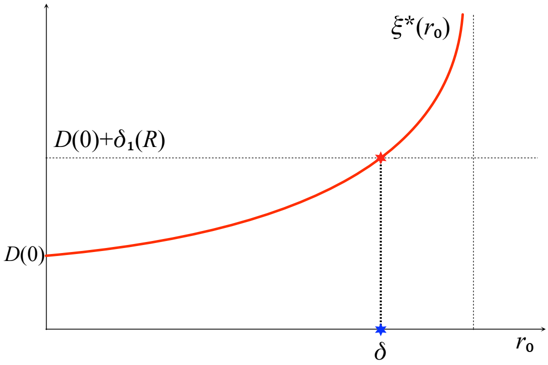
Four possibilities that may arise with closed-loop solutions.

Proof (1)



The strategy of the proof. The exact time σ^* when the control reaches the plant is not known analytically. We find an upper bound $\xi^* \geq \sigma^*$ by using an upper bound $D(X) \leq D(0) + \delta_1(|X|)$ on the delay and by estimating an upper bound on the open-loop solution $|X(t)| \leq \eta(r_0, t)$, $r_0 := |X(0)| + \sup_{-D(X(0)) \leq \theta \leq 0} |U(\theta)|$.

Proof (2)

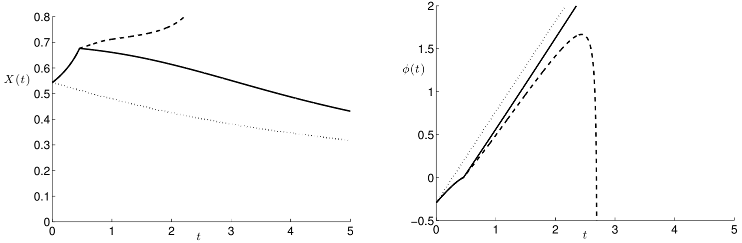


The function $\xi^*(r_0)$ determined from the fixed-point problem $\xi^* = D(0) + \delta_1(\eta(r_0, \xi^*)$. By reducing r_0 sufficiently, we can ensure that the control signal reaches the plant before $|X(t)|$ has exceeded R.

A Locally Stabilizable Example

$$\dot{X}(t) = X(t)^4 + 2X(t)^5 + \left(X(t)^2 + X(t)^3\right) U\left(\overbrace{t - X(t)^2}^{\phi(t)}\right)$$

not locally exponentially stabilizable nor globally asymptotically stabilizable. Delay-free controller $U(t) = -X(t)$ yields $\dot{X}(t) = -X(t)^3 + 2X(t)^5$, with $R = \frac{1}{\sqrt{2}}$.



Solid: Delay-compensating controller. Dashed: Uncompensated controller. Dot: Nominal controller for a system without delay. The controller "kicks in" at $\sigma^* = 0.46$ and hence, $X^* = \sqrt{\sigma^*} = 0.678$, which is almost at $R = \frac{1}{\sqrt{2}}$.

Nonlinear Predictor Feedback for Simultaneous Time-Varying Delays on the Input and the State

Nonlinear Systems with Delayed Integrators (Design)

$$\begin{aligned} \dot{X}_1(t) &= f_1\left(X_1(t)\right) + X_2\left(\phi_1(t)\right) \\ \dot{X}_2(t) &= f_2\left(X_1(t), X_2(t)\right) + U\left(\phi_2(t)\right) \end{aligned}$$

Predictor feedback law:

$$\begin{aligned} U(t) &= -f_2\left(P_1\left(\psi\left(\phi_2^{-1}(t)\right)\right), P_2(t)\right) - c_2\left(P_2(t) + c_1 P_1(t) + f_1\left(P_1(t)\right)\right) \\ &\quad - \left(c_1 + \frac{\partial f_1\left(P_1\right)}{\partial P_1}\right)\left(f_1\left(P_1(t)\right) + P_2(t)\right) R\left(\phi_2^{-1}(t)\right) \end{aligned}$$

Predictor formula:

$$\begin{aligned} P_1(t) &= X_1(t) + \int_{\psi(t)}^{t} \left(f_1\left(P_1(\theta)\right) + P_2(\theta)\right) \frac{d\theta}{\psi'\left(\psi^{-1}(\theta)\right)} \\ P_2(t) &= X_2(t) + \int_{\phi_2(t)}^{t} \frac{\left(f_2\left(P_1\left(\psi\left(\phi_2^{-1}(\theta)\right)\right), P_2(\theta)\right) + U(\theta)\right) d\theta}{\phi_2'\left(\phi_2^{-1}(\theta)\right)} \\ \psi(t) &= \phi_2\left(\phi_1(t)\right) \end{aligned}$$

In the constant delay case:

$\phi_i(t) = t - D_i, \quad \psi(t) = \phi_2\left(\phi_1(t)\right) = t - D_1 - D_2$

Assumptions

Plant

$$\begin{aligned}\dot{X}_1(t) &= f_1\left(X_1(t)\right) + X_2\left(\phi_1(t)\right)\\ \dot{X}_2(t) &= f_2\left(X_1(t), X_2(t)\right) + U\left(\phi_2(t)\right)\end{aligned}$$

is **forward complete**.

Delays

$D_i(t)$ **positive and bounded**.

$\dot{D}_i(t)$ **less than one**.

Nonlinear Systems with Delayed Integrators (Analysis)

Backstepping Transformation:

$$Z_2(t) = X_2(t) + f_1\left(P_1\left(\phi_2(t)\right)\right) + c_1 P_1\left(\phi_2(t)\right)$$

Traget system:

$$\begin{aligned} \dot{Z}_1(t) &= -c_1 Z_1(t) + Z_2(\phi_1(t)) \\ \dot{Z}_2(t) &= -c_2 Z_2(t) + W(\phi_2(t)) \\ W(t) &= 0, \quad t \geq 0 \end{aligned}$$

Theorem 6:

$\exists \hat{\beta} \in \mathcal{KL}$ such that

$$|X_1(t)| + \|X_2(t)\|_\infty + \|U(t)\|_\infty \leq \hat{\beta}\left(|X_1(0)| + \|X_2(0)\|_\infty + \|U(0)\|_\infty, t\right)$$

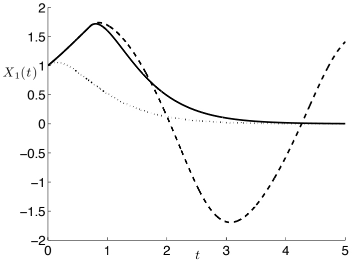
$\|\cdot\|_\infty$ sup norms over $[t - D_i(t), t]$

An Example

$$\dot{X}_1(t) = \sin\left(X_1(t)\right) + X_2\left(t - D(t)\right), \quad \dot{X}_2(t) = U(t)$$

Predictor feedback

$$\begin{aligned} U(t) &= -c_2\left(X_2(t) + c_1 P_1(t) + \sin\left(P_1(t)\right)\right) \\ &\quad - \left(c_1 + \cos\left(P_1(t)\right)\right)\left(\sin\left(P_1(t)\right) + X_2(t)\right) \frac{d(t - D(t))^{-1}}{dt} \\ P_1(t) &= X_1(t) + \int_{\underbrace{t - D(t)}_{\phi(t)}}^{t} \left(\sin\left(P_1(\theta)\right) + X_2(\theta)\right) \frac{d\theta}{\phi'\left(\phi^{-1}(\theta)\right)} \end{aligned}$$



- Dotted: Assuming $D(t) = 0$ and $D(t) = 0$
- Dashed: Assuming $D(t) = 0$ but $D(t) = \frac{1+t}{1+2t}$
- Solid: With predictor feedback
- Control signal reaches $X_1(t)$ at $t = \phi^{-1}(0) = \frac{1}{\sqrt{2}}$

State-Dependent State Delay

Nonlinear Systems with State-Dependent State Delay

$$\begin{aligned}\dot{X}_1(t) &= f_1\left(t, X_1(t), X_2\left(t - D\left(X_1(t)\right)\right)\right) \\ \dot{X}_2(t) &= f_2\left(t, X_1(t), X_2(t)\right) + U(t)\end{aligned}$$

Additional challenge: The predictor design does not follow immediately from the delay-free design

Predictor Feedback

$$U(t) = -f_2(t, X_1(t), X_2(t)) - c_2(X_2(t) - \kappa(\sigma(t), P_1(t))) + \frac{\frac{\partial \kappa(\sigma, P_1)}{\partial \sigma} + \frac{\partial \kappa(\sigma, P_1)}{\partial P_1} f_1(\sigma(t), P_1(t), X_2(t))}{1 - \nabla D(P_1(t)) f_1(\sigma(t), P_1(t), X_2(t))},$$

where

$$P_1(t) = X_1(t) + \int_{t-D(X_1(t))}^{t} \frac{f_1(\sigma(s), P_1(s), X_2(s))\, ds}{1 - \nabla D(P_1(s)) f_1(\sigma(s), P_1(s), X_2(s))}$$

$$\sigma(t) = t + D(P_1(t))$$

Why the Results are not Global (Even for Forward Complete Systems)

The feasibility condition that the *delay rate is less than one* must hold to ensure that the control signal reaches the state X_1 and that the control remains bounded

The solutions of the system and the initial conditions must satisfy

$$\mathcal{G}_c : \qquad \nabla D\left(P_1(\theta)\right) f_1\left(\sigma(\theta), P_1(\theta), X_2(\theta)\right) < c,$$

for all $\theta \geq t_0 - D\left(X_1(t_0)\right)$, for $c \in (0, 1]$. We refer to $\mathcal{G}_1$ as the *feasibility condition* of the controller.

Theorem 7:

$\exists \xi_{\mathrm{RoA}} \in \mathcal{K}$ and $\beta^* \in \mathcal{KL}$ such that for all initial conditions that satisfy

$$\begin{aligned} \Omega(t_0) &< \xi_{\mathrm{RoA}}(c) \\ \Omega(t) &= |X_1(t)| + \sup_{t-D(X_1(t))\leq\theta\leq t} |X_2(\theta)| \end{aligned}$$

for some $0 < c < 1$,

$$\Omega(t) \leq \beta^*(\Omega(t_0), t - t_0)$$

Assumptions (Delay-Free Plant)

$\exists$ smooth positive definite function R and α_1, α_2 and $\alpha_3 \in \mathcal{K}_\infty$ s.t. $\forall (X, \omega, t)$

$$\begin{aligned} \alpha_1(|X|) &\leq R(t,X) \leq \alpha_2(|X|) \\ \frac{\partial R(t,X)}{\partial t} + \frac{\partial R(t,X)}{\partial X} f_1(t,X,\omega) &\leq R(t,X) + \alpha_3(|\omega|), \end{aligned}$$

which guarantees that $\dot{X} = f_1(t, X, \omega)$ is forward-complete.

$\dot{X}(t) = f_1(t, X(t), \kappa(t, X(t)) + \omega(t))$ is ISS

Assumptions (Delay)

$D \in C^1 \left(\mathbb{R}^n; \mathbb{R}_+\right)$

Proof (Infinite-Dimensional Backstepping Transformation)

The infinite-dimensional backstepping transformation of the state X_2

$$Z_2(\theta) = X_2(\theta) - \kappa\left(\theta + D\left(P_1(\theta)\right), P_1(\theta)\right), \quad t - D\left(X_1(t)\right) \leq \theta \leq t$$

transforms the system to the "target system"

$$\begin{aligned}
\dot{X}_1(t) &= f_1\left(t, X_1(t), \kappa\left(t, X_1(t)\right) + Z_2\left(t - D\left(X_1(t)\right)\right)\right) \\
\dot{Z}_2(t) &= -c_2 Z_2(t)
\end{aligned}$$

Then prove stability of "target system" from ISS.

Using forward-completeness and ISS prove the norm equivalency.

On the way to do so, prove that the *feasibility condition* is satisfied when the original norm is small.

Nonlinear Systems with State-Dependent State Delay (Example 1)

$$\begin{aligned}\dot{s}(t) &= v(t - r_1 \sin^2(\omega s(t))) \\ \dot{v}(t) &= a(t)\end{aligned}$$

The predictor-based controller is

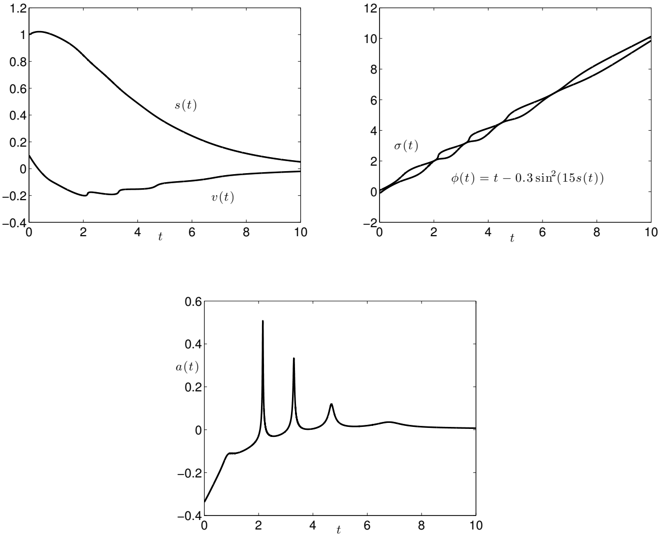
$$a(t) = -c_2 \left(v(t) + c_1 P_1(t)\right) - c_1 \frac{v(t)}{1 - r_1 \omega \sin\left(\omega P_1(t)\right) \cos\left(\omega P_1(t)\right) v\left(t\right)}$$

where

$$P_1(t) = s(t) + \int_{t - r_1 \sin^2(\omega s(t))}^{t} \frac{v\left(s\right) ds}{1 - r_1 \omega \sin\left(\omega P_1(s)\right) \cos\left(\omega P_1(s)\right) v\left(s\right)}$$

Nonlinear Systems with State-Dependent State Delay (Example 1)

The initial conditions are $s(0) = 1$, $v(\theta) = 0.1$, for all $-r_1 \sin^2(\omega s(0)) \leq \theta \leq 0$



Nonlinear Systems with State-Dependent State Delay (Example 2)

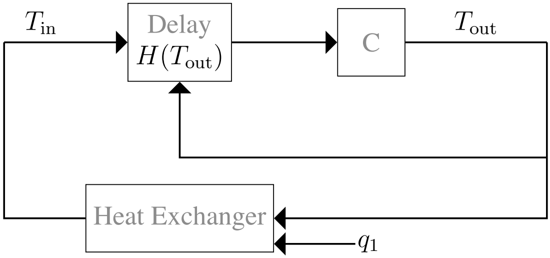


Figure: A marine cooling system with one consumer

$T_{\text{out}} = X_1$, $T_{\text{in}} = X_2$, $H = \frac{b}{k_1 T_{\text{out}} + k_2} = D$, $q_1 = U$

$$\begin{aligned} \dot{X}_1(t) &= a\left(X_1(t) - X_2\left(t - D(X_1(t))\right)\right)\left(k_1 X_1(t) + k_2\right) \\ \dot{X}_2(t) &= \left(k_1 X_1(t) + k_2\right)\left(X_1(t) - X_2(t)\right) - U(t) \end{aligned}$$

where, $a < 0$, $b, k_1, k_2 > 0$.

Nonlinear Systems with State-Dependent State Delay (Example 2)

We choose

$$
\begin{aligned}
U(t) \;&=\; (k_1 X_1(t) + k_2)\,(X_1(t) - X_2(t)) + c_2\left(X_2(t) - P_1(t) - \frac{c_1}{a}\frac{P_1(t) - T_{\mathrm{eq}}}{k_1 P_1(t) + k_2}\right) \\
&\quad - \left(1 + \frac{c_1}{a k_1}\frac{T_{\mathrm{eq}} + \frac{k_2}{k_1}}{\left(P_1(t) + \frac{k_2}{k_1}\right)^2}\right) \times \frac{(P_1(t) - X_2(t))\,(k_1 P_1(t) + k_2)}{R(t)}
\end{aligned}
$$

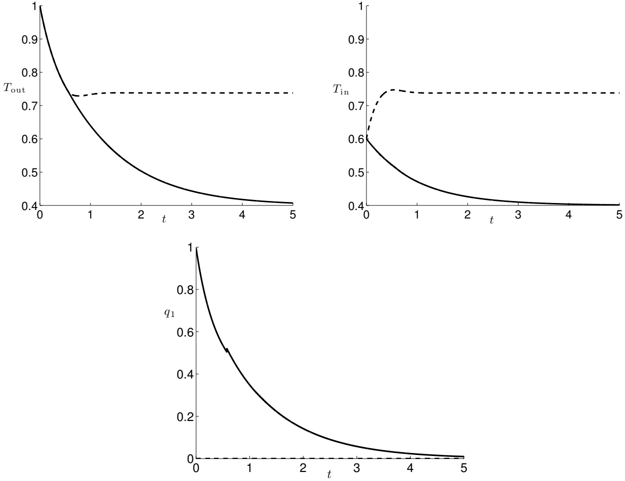
where

$$
\begin{aligned}
P_1(t) &= X_1(t) + \int_{t-\frac{b}{k_1 X_1(t) + k_2}}^{t} a\frac{(P_1(\theta) - X_2(\theta))\,(k_1 P_1(\theta) + k_2)\,d\theta}{R(\theta)} \\
R(\theta) &= 1 + \frac{b k_1 a\,(P_1(\theta) - X_2(\theta))}{(k_1 P_1(\theta) + k_2)^2}\,(k_1 P_1(\theta) + k_2)
\end{aligned}
$$

Nonlinear Systems with State-Dependent State Delay (Example 2)

$a = -1$, $c_1 = c_2 = b = k_1 = k_2 = 1$, $T_{\mathrm{out}}(0) = 1$, $T_{\mathrm{in}}(\theta) = 0.6, \theta \leq 0$, $T_{\mathrm{eq}} = 0.4$

Figure: Dashed: Open-loop response. Solid: Response with predictor feedback.



Robustness of Linear Constant-Delay Predictors to Time-Varying Delay Perturbations

Robustness to Time-Varying Delay Perturbations

$$\dot{X}(t) = AX(t) + BU\left(t - \hat{D} - \delta(t)\right)$$

$$U(t) = K\left(e^{A\hat{D}}X(t) + \int_{t-\hat{D}}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right)$$

Theorem 8: $\exists\ \delta_1$, such that if

$$|\delta(t)| + |\delta'(t)| < \delta_1, \quad \text{for all } t \geq 0$$

then, the closed-loop system is exponentially stable, in the sense of the norm

$$\Pi_{\mathrm{L}}(t) = |X(t)|^2 + \int_{t-\hat{D}-\max\{0,\delta(t)\}}^{t} U(\theta)^2 d\theta + \int_{t-\hat{D}}^{t} \dot{U}(\theta)^2 d\theta$$

Robustness to Time-Varying Delay Perturbations

Are larger absolute delay and delay rates allowed?

Theorem 9: $\exists$ δ_2, δ_3 such that if

$$\int_0^\infty \left(|\delta'(\theta)| + |\delta(\theta)|\right) d\theta \quad \leq \quad \delta_2$$

or

$$|\delta(t)| + |\delta'(t)| \quad \to \quad 0, \quad \text{when } t \to \infty$$

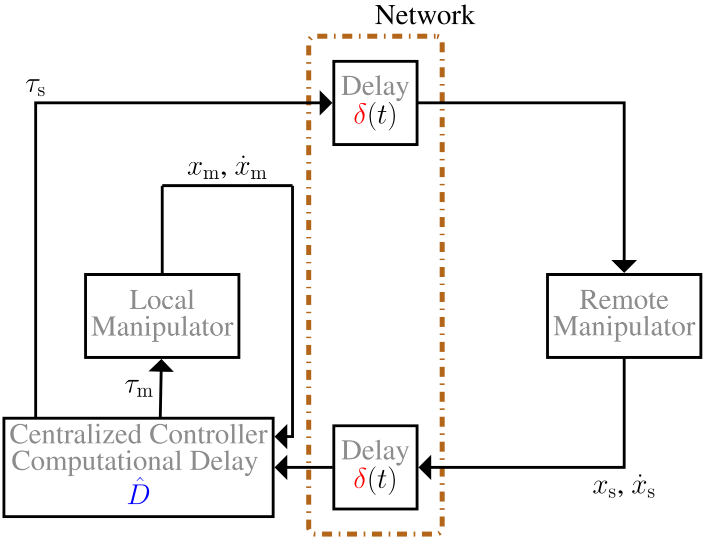
or

$$\frac{1}{\Delta} \int_t^{t+\Delta} \left(|\delta'(\theta)| + |\delta(\theta)|\right) d\theta \quad \leq \quad \delta_3 \quad \text{for all } t \geq T$$

then, the closed-loop system is exponentially stable, in the sense of the norm

$$\Pi_{\mathrm{L}}(t) \quad = \quad |X(t)|^2 + \int_{t-\hat{D}-\max\{0,\delta(t)\}}^{t} U(\theta)^2 d\theta + \int_{t-\hat{D}}^{t} \dot{U}(\theta)^2 d\theta$$

Centralized Teleoperation



$$
\begin{aligned}
\ddot{x}_m(t) + \dot{x}_m(t) &= \tau_m\left(t - \hat{D} - \delta(t)\right) \\
\ddot{x}_s(t) + \dot{x}_s(t) &= \tau_s\left(t - \hat{D} - 2\delta(t)\right)
\end{aligned}
$$

Centralized Teleoperation (Design)

$$U_1(t) = -K_\mathrm{p}\left(\hat{P}_1(t) - \hat{P}_3(t)\right) - B_\mathrm{m}\hat{P}_2(t) - K_\mathrm{p}\left(\hat{P}_1(t) - r\right)$$

$$U_2(t) = K_\mathrm{p}\left(\hat{P}_1(t) - \hat{P}_3(t)\right) - B_\mathrm{s}\hat{P}_4(t) - K_\mathrm{p}\left(\hat{P}_3(t) - r\right)$$

$$\hat{P}_i(t) = X_i(t) + \int_{t-\hat{D}}^{t} \hat{P}_{i+1}(\theta)d\theta, \quad i = 1, 3$$

$$\hat{P}_j(t) = X_j(t) + \int_{t-\hat{D}}^{t} \left(-\hat{P}_j(\theta) + U_{\frac{j}{2}}(\theta)\right)d\theta, \quad j = 2, 4$$

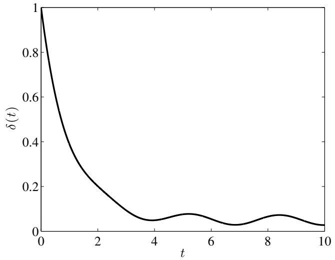


Figure: The delay perturbation δ induced by the network in teleoperation.

Centralized Teleoperation (Simulations)

$\hat{D} = 1$

$\dot{\delta}(t) = -\delta(t) + 0.1\sin(t)^2$, $\delta(0) = 1$ **(solid)**. $\delta(t) = 0$ **(dashed)**.

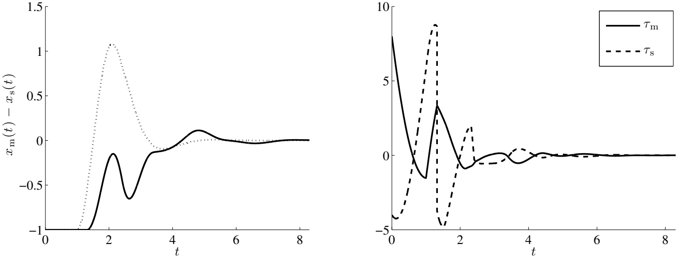


Figure: The error between the position of the master and the slave and the input torques. The two robots are coordinated through a network that induces an unknown time-varying delay $\delta(t)$. The initial conditions are $x_{\mathrm{m}}(0) = 0$, $x_{\mathrm{s}}(0) = 1$, $\dot{x}_{\mathrm{m}}(0) = \dot{x}_{\mathrm{s}}(0) = 0$, $\tau_{\mathrm{m}}(\theta) = \tau_{\mathrm{s}}(\theta) = 0$, $-1 - \delta(0) \leq \theta \leq 0$.

Robustness of Nonlinear Constant-Delay Predictors to Time- and State-Dependent Delay Perturbations

Motivation

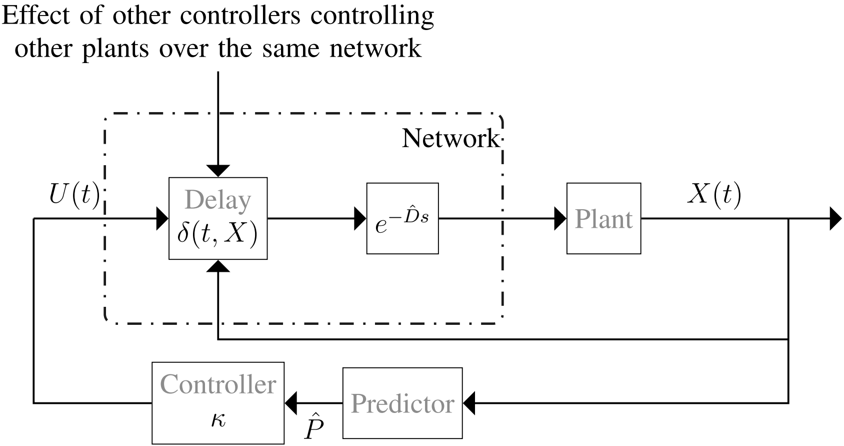


Figure: Control over a network, with delay that varies with time (as a result of other users's activities) and may be state-dependent. The designer only knows a nominal, constant delay value $\hat{D}$. The delay fluctuation $\delta(t, X)$ is unknown.

Robustness to Time- and State-Dependent Delay Perturbations

$$
\begin{aligned}
\dot{X}(t) &= f\left(X(t), U\left(t - \hat{D} - \delta\left(t, X(t)\right)\right)\right) \\
U(t) &= \kappa\left(\hat{P}(t)\right) \\
\hat{P}(t) &= X(t) + \int_{t-\hat{D}}^{t} f\left(\hat{P}(s), U(s)\right) ds
\end{aligned}
$$

Assumptions

$\dot{X} = f\left(X, \omega\right)$ **is forward-complete**

$\dot{X} = f\left(X, \kappa(X)\right)$ is **l.e.s**.

Theorem 10: $\exists\ c_1,\ c^{**} > 0$, and $\hat{\mu},\ \alpha^*,\ \zeta \in \mathcal{K}_\infty$, and $\beta \in \mathcal{KL}$ such that if

$$|\delta(t,\xi)| + |\delta_t(t,\xi)| + |\nabla\delta(t,\xi)| \leq c_1 + \hat{\mu}(|\xi|)$$

then for all

$$\Pi(0) < c^{**}$$

it holds

$$\Pi(t) \leq \beta(\Pi(0), t), \quad \text{for all } t \geq 0$$

where

$$\begin{aligned}\Pi(t) &= |X(t)| + \int_{t-\hat{D}}^{t} \alpha^*(|U(\theta)|)\,d\theta + \int_{t-\hat{D}-\max\{0,\delta(t,X(t))\}}^{t} \dot{U}(\theta)^2 d\theta \\ &\quad + \int_{t-\hat{D}}^{t} \ddot{U}(\theta)^2 d\theta\end{aligned}$$

Control of a DC Motor over a Network (1)

$$\begin{aligned}
\frac{d\omega(t)}{dt} &= \theta i_{\mathrm{f}}(t) i_{\mathrm{a}}(t) \\
\frac{di_{\mathrm{a}}(t)}{dt} &= -b i_{\mathrm{a}}(t) + k - c i_{\mathrm{f}}(t)\omega(t) \\
\frac{di_{\mathrm{f}}(t)}{dt} &= -a i_{\mathrm{f}}(t) + U\left(t - \hat{D} - \delta(t, i_{\mathrm{f}}(t), i_{\mathrm{a}}(t), \omega(t))\right)
\end{aligned}$$

i_{f}, i_{a} are field and armature currents and ω is angular velocity.
Delay-free design (based on full-state linearization)

$$\begin{aligned}
U(t) &= 1/\gamma \times (-K_1 Z_1(t) - K_2 Z_2(t) - K_3 Z_3(t) - \alpha) \\
Z_1(t) &= \theta i_{\mathrm{a}}(t)^2 + c\omega(t)^2 - \theta \frac{k^2}{b^2} - c\omega_0^2 \\
Z_2(t) &= 2\theta i_{\mathrm{a}}(t)\left(k - b i_{\mathrm{a}}(t)\right) \\
Z_3(t) &= 2\theta\left(k - 2b i_{\mathrm{a}}(t)\right)\left(-b i_{\mathrm{a}}(t) + k - c i_{\mathrm{f}}(t)\omega(t)\right) \\
\gamma &= -2c\theta\left(k - 2b i_{\mathrm{a}}(t)\right)\omega(t) \\
\alpha &= 2ca\theta\left(k - 2b i_{\mathrm{a}}(t)\right) i_{\mathrm{f}}(t)\omega(t) - 2b\theta\left(3k - 4b i_{\mathrm{a}}(t)\right. \\
&\quad \left. - 2c i_{\mathrm{f}}(t)\omega(t)\right)\left(-b i_{\mathrm{a}}(t) + k - c i_{\mathrm{f}}(t)\omega(t)\right) \\
&\quad - 2c\theta\left(k - 2b i_{\mathrm{a}}(t)\right) i_{\mathrm{f}}(t)^2 \omega(t).
\end{aligned}$$

Control of a DC Motor over a Network (2)

Nominal delay: $\hat{D} = 1$
Delay Perturbation: $\delta(t, i_{\mathrm{a}}(t)) = 0.5 i_{\mathrm{a}}(t)^2 + 0.2\sin(t)^2$ (**solid**). $\delta(t, i_{\mathrm{a}}(t)) = 0$ (**dashed**).

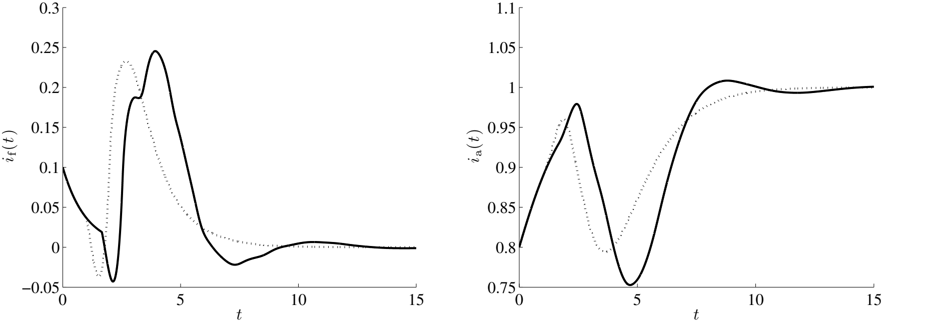


Figure: The field and armature currents with initial conditions $i_{\mathrm{f}}(0) = 0.1$, $i_{\mathrm{a}}(0) = 0.8$.

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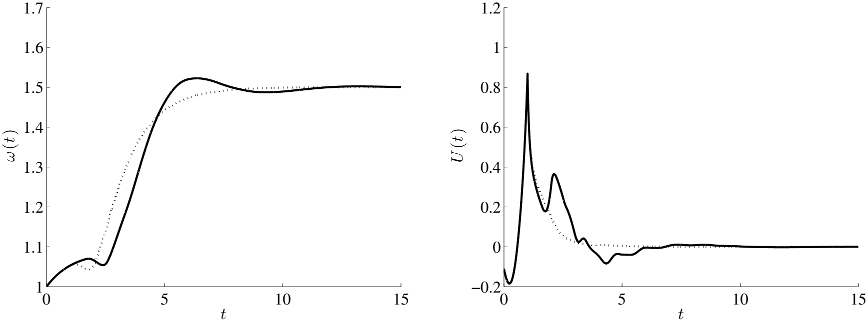


Figure: The angular velocity and the field voltage. The initial conditions are $\omega(0) = 1$ and $U(\theta) = 0$, $-1 - \delta(0, i_{\mathrm{a}}(0)) \leq \theta \leq 0$.

For the Future

- Robust control for linear systems with constant input delays
- Nonlinear systems with constant distributed input delays
- Nonlinear systems with more complex input dynamics
- State-dependent delays that depend on delayed states
- Input-dependent delays

Sampled-data and control over networks

(with Iasson Karafyllis)

Sampled-data stabilization of LTI systems with input delay τ, measurement delay r, and sampling time T

Theorem 11: *Let* $l = \mathrm{integer}\left\{\frac{\tau + r}{T}\right\} \in \mathbb{Z}$ *and* $\delta = \tau + r - lT$. *Suppose that the matrix*

$$\exp(AT)\left(I + \int_0^T \exp(-As)ds BK\right)$$

has all of its eigenvalues inside the unit circle (guaranteed for suffic. small T). The controller

$$u(t) = u_i\,, \qquad t \in [iT, (i+1)T)\,, \quad i \in \mathbb{Z}^+$$

with input applied with zero-order hold given by

$$u_i = K\exp(A(\tau + r))x(iT - r) + K\sum_{j=1}^{l+1} Q_j B u_{i-j}\,, \qquad \text{[difference eqn in } u\text{]}$$

where

$$Q_j = \exp(AjT)\int_0^T \exp(-As)ds\,, \qquad j = 1, \ldots, l$$

$$Q_{l+1} = \exp(AlT)\int_0^\delta \exp(As)ds$$

guarantees exp. stability in the supremum norm of x *over* $[-r, 0]$ *and* u *over* $[-\tau, 0]$.

Nonlinear Predictor Fbk for Non-holonomic Unicycle Controlled Over a Network with Arbitrarily Sparse Sampling

Let (for simplicity)

$$D = \text{transmission delay in both directions} = \text{sampling time}$$

Controller

$$v(t) = \frac{1}{D}\left(k_1(P,Q,\Theta) + Qk_2(P,Q,\Theta)\right), \quad \text{for} \quad t \in [iD,(i+1)D)$$

$$\omega(t) = \frac{1}{D}k_2(P,Q,\Theta), \quad \text{for} \quad t \in [iD,(i+1)D)$$

with transformation

$$P = X\cos(\Theta) + Y\sin(\Theta)$$

$$Q = X\sin(\Theta) - Y\cos(\Theta)$$

the exact predictor of $(x,y,\theta)((i+1)D)$

$$X = x((i-1)D) + \int_{(i-2)D}^{iD} v(s)\cos\left(\theta((i-1)D) + \int_{(i-2)D}^{s}\omega(z)dz\right)ds$$

$$Y = y((i-1)D) + \int_{(i-2)D}^{iD} v(s)\sin\left(\theta((i-1)D) + \int_{(i-2)D}^{s}\omega(z)dz\right)ds$$

$$\Theta = \theta((i-1)D) + \int_{(i-2)D}^{iD}\omega(s)ds$$

and the discontinuous sampled-data stabilizer designed for the delay-free case

$$k_1(P,Q,\Theta) = -\begin{cases} |Q|^{1/2}, & Q(2Q-P\Theta)\neq 0 \\ \dfrac{P^2\Theta}{P^2+\Theta^2}, & Q=0, P\Theta\neq 0 \\ \Theta, & 2Q=P\Theta \end{cases}$$

$$k_2(P,Q,\Theta) = -\begin{cases} 2\left(P+\operatorname{sgn}(Q)|Q|^{1/2}\right), & Q(2Q-P\Theta)\neq 0 \\ \dfrac{P\Theta^2}{P^2+\Theta^2}, & Q=0, P\Theta\neq 0 \\ P, & 2Q=P\Theta \end{cases}$$

Theorem 12: *For any $D>0$, the closed-loop system is globally asymptotically stable at the origin. Moreover, $x(t)=y(t)=\theta(t)=0$ for $t\geq 5D$.*

Predictor allows stabilization based on states from two long sample periods ago.

Adaptive Control for **Unknown Delay**

Robustness to Delay Mismatch

The biggest open question in robustness of predictor feedbacks.

$$\begin{aligned}\dot{X} &= AX + BU(t - D_0 - \Delta D) \\ U(t) &= K\left[\mathrm{e}^{AD_0}X(t) + \int_{t-D_0}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right]\end{aligned}$$

ΔD either positive or negative

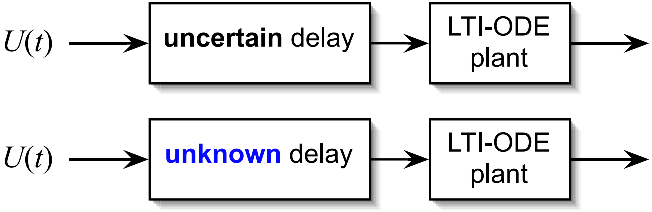
Theorem 13: $\exists \delta > 0$ *s.t.* $\forall \Delta D \in (-\delta, \delta)$ *the closed-loop system is exp. stable in the sense of the state norm*

$$N_2(t) = \left(|X(t)|^2 + \int_{t-\bar{D}}^{t} U(\theta)^2 d\theta\right)^{1/2},$$

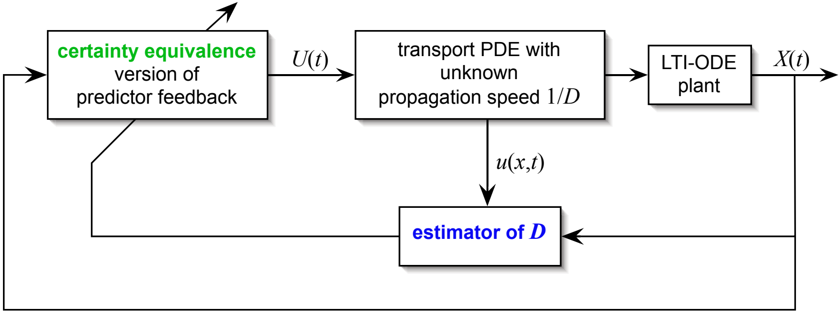
where $\bar{D} = D_0 + \max\{0, \Delta D\}$.

Delay-Robustness of Predictor Feedback

Delay-Adaptive Control



Motivation: control of thermoacoustic instabilities in gas turbine combustors


certainty equivalence
version of
predictor feedback
U(t)
transport PDE with
unknown
propagation speed 1/D
LTI-ODE
plant
X(t)
u(x,t)
estimator of D

Update law

$$\frac{d}{dt}\hat{D}(t) = -\gamma \frac{\int_0^1 (1+x) \overbrace{w(x,t)}^{\text{reg. error}} \overbrace{K e^{A\hat{D}(t)x} dx \quad (AX(t) + Bu(0,t))}^{\text{regressor}}}{\underbrace{1 + X(t)^T P X(t) + b \int_0^1 (1+x) w(x,t)^2 dx}_{\text{normalization}}}$$

$$w(x,t) = u(x,t) - \hat{D}(t) \int_0^x K e^{A\hat{D}(t)(x-y)} Bu(y,t) dy - K e^{A\hat{D}(t)x} X(t) \, .$$

Update law

$$\frac{d}{dt}\hat{D}(t) = -\gamma \frac{\int_0^1 (1+x) \overbrace{w(x,t)}^{\text{reg. error}} \overbrace{Ke^{A\hat{D}(t)x}dx \ (AX(t)+Bu(0,t))}^{\text{regressor}}}{\underbrace{1+X(t)^T PX(t) + b\int_0^1 (1+x)w(x,t)^2 dx}_{\text{normalization}}}$$

$$w(x,t) = u(x,t) - \hat{D}(t)\int_0^x Ke^{A\hat{D}(t)(x-y)}Bu(y,t)dy - Ke^{A\hat{D}(t)x}X(t)\,.$$

Theorem 14: $\exists R, \rho > 0$ *s.t.*

$$\Upsilon(t) \le R\left[\exp\left(\rho\Upsilon(0)\right) - 1\right]$$ (exp. growing class $\mathcal{K}_\infty$ glob. stab. bound)

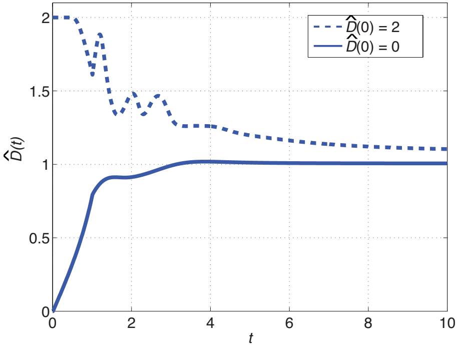
where

$$\Upsilon(t) = |X(t)|^2 + \int_0^1 u(x,t)^2 dx + \left(D - \hat{D}(t)\right)^2.$$

Furthermore, $X(t), U(t) \to 0$.

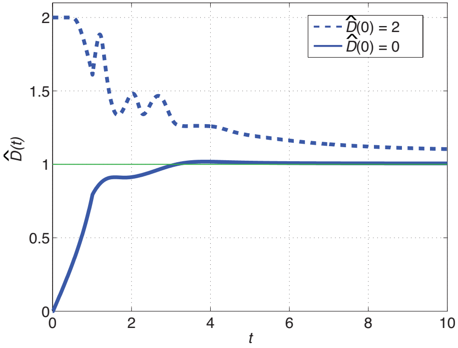
$$X(s) = \frac{e^{-s}}{s - 0.75} U(s)$$

unstable X-29 aircraft
[Ens, Ozbay, Tannenbaum, 1992]



$$X(s) = \frac{e^{-s}}{s - 0.75} U(s)$$

Simulations by
Delphine Bresch-Pietri



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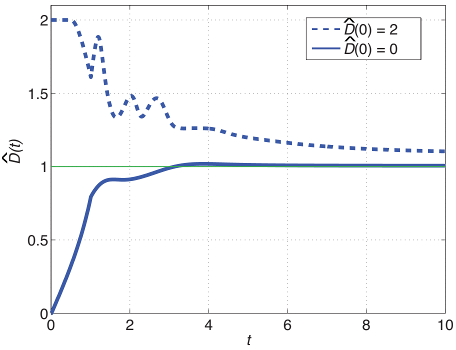

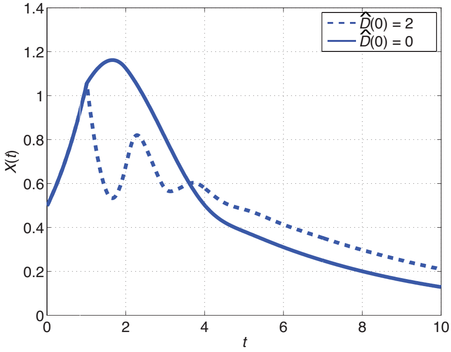

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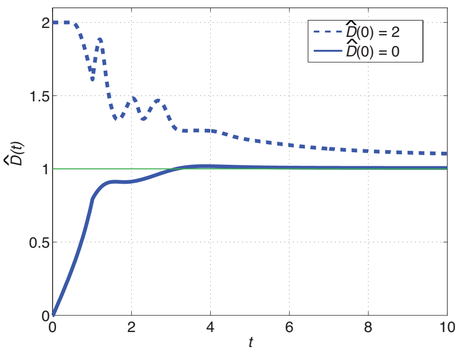

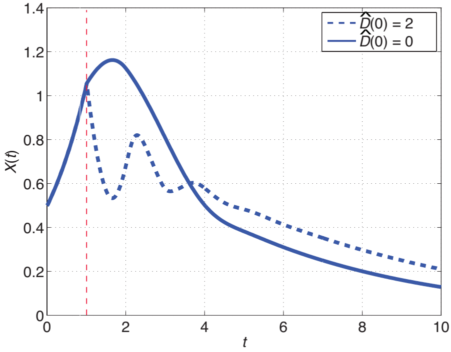

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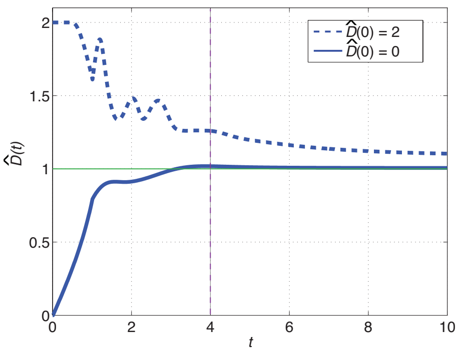

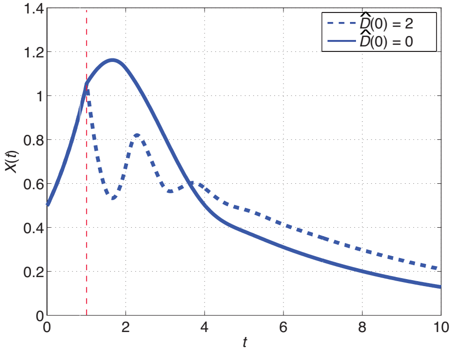

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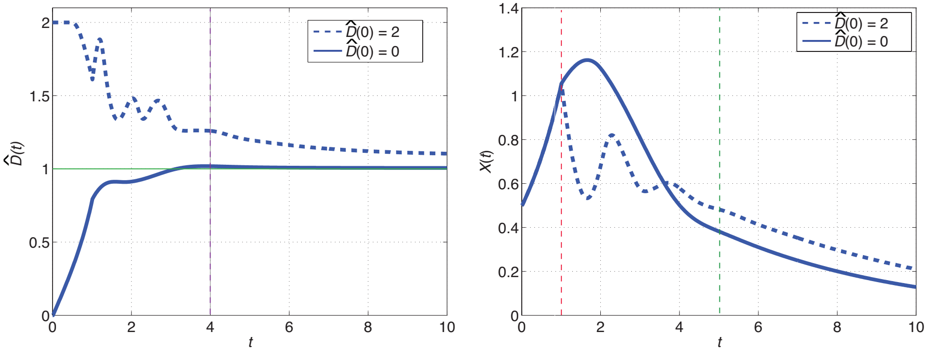

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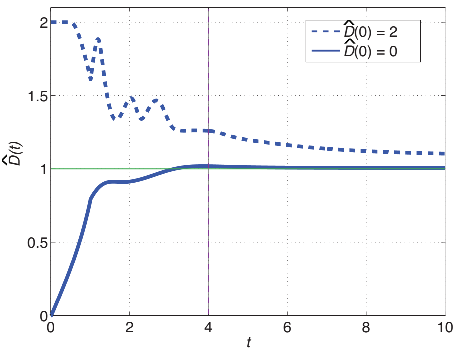

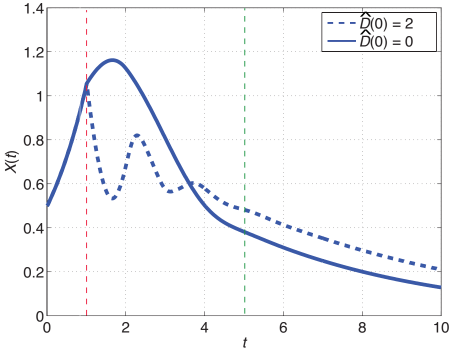

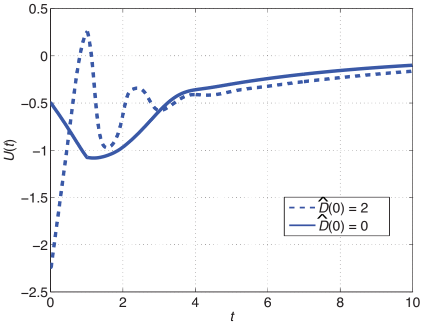

$$X(s) = \frac{e^{-s}}{s - 0.75} U(s)$$

Simulations by
Delphine Bresch-Pietri

0–1 sec The delay precludes any influence of the control on the plant, so $X(t)$ shows an exponential open-loop growth.

1–3 sec The plant starts responding to the control and its evolution changes qualitatively, resulting also in a qualitative change of the control signal.

3–4 sec When the estimation of $\hat{D}(t)$ ends at about 3 seconds, the controller structure becomes linear. However, due to the delay, the plant state $X(t)$ continues to evolve based on the inputs from 1 second earlier, so, a non-monotonic transient continues until about 4 seconds.

4 sec and onwards The (X, U) system is linear and the delay is sufficiently well compensated, so the response of $X(t)$ and $U(t)$ shows a monotonically decaying exponential trend of a first order system.