FEEDBACK CONTROL OF SCALAR CONSERVATION LAWS WITH APPLICATION TO DENSITY CONTROL IN FREEWAYS BY MEANS OF VARIABLE SPEED LIMITS

Iasson Karafyllis and Markos Papageorgiou

Speed limits may vary…

Speed limits may be very small…

How can we set appropriate speed limits?

Outline

➔ The model

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- ➔ The model
- ➔ Two Control Problems

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- ➔ Two Possible Solutions

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- ➔ Example
- ➔ Future Research

The model

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(the standard LWR model)

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(the “mean” driver respects the law)

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This is the era of automated and connected vehicles!

The traffic control centre may communicate different variable speed limits to individual equipped vehicles at *virtually arbitrary* space, time and value resolution

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Example: In Carlson, R. C., *Mainstream Traffic Flow Control on Motorways*, Ph.D. Thesis, Technical University of Crete, Chania, Greece, 2011

$$F(\rho, l) = A\rho l \exp\left(-\frac{1}{\gamma}\left(\frac{b\rho}{1+a-al}\right)^{\gamma}\right)$$

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Define:

$$f(\rho) := F(\rho, 1), \text{ for } \rho \in [0, \rho_{\max}]$$

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(the flow-density curve in absence of speed limits)

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In general:

(H) *The function* $f \in C^2\left([0,\rho_{\max}];[0,q_{\max}]\right)$*, where* $q_{\max} > 0$ *is a function for which there exists* $\rho_{cr} \in (0,\rho_{\max})$ *with the following properties:*

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(iii) $f(0) = 0$ *and* $f(\rho) > 0$ *for all* $\rho \in (0, \rho_{\max}]$.

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(standard assumption for the fundamental diagram)

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A major simplification:

For every $u \in (0,1]$, $\rho \in \left(0, \rho_{\max}\right]$ there exists $l \in (0,1]$ such that

$$u\, f(\rho) = F(\rho, l)$$

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Construct a feedback law

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with

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so that

$$\lim_{t\to+\infty}\big(\rho(t,x)\big) = \rho^*, \ \lim_{t\to+\infty}\big(u(t,x)\big) = 1, \text{ for } x \in [0,L].$$

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the freeway will practically operate without speed limits
after an initial transient period,
i.e. after the problem has been tackled.

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queues may be created at the entrance of the freeway

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$\rho(t,0) = \rho^*$, for $t \geq 0$ (boundary condition)

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Both Control Problems are Nonlinear

Two Possible Solutions

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the initial value problem with $\rho[0] = \rho_0$ *and*

$$u(t,x) = \frac{\min\limits_{z\in[0,L]}\left(f(\rho(t,z)) M\left(\rho[t], z\right)\right)}{f(\rho(t,x)) M\left(\rho[t], x\right)}, \text{ for } t \geq 0,\ x \in [0,L]$$

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has a unique solution $\rho \in C^1\left(\Re_+ \times [0,L];(0,\rho_{\max}]\right)$, *which satisfies:*

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where $c=c\left(\min\limits_{z\in[0,L]}\left(\rho_0(z)\right)\right)$.

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Then there exist constants $\sigma, \gamma, \omega > 0$ *such that for every* $\rho_0 \in C^1\left([0,L];(0,\rho_{\max}]\right)$ *with* $\rho_0(0) = \rho^*$ *and*

$$f\left(\rho^*\right) + \sigma \int_0^x \left(\rho_0(s) - \rho^*\right) ds - \gamma \frac{x^2}{2} \left\| \rho_0 - \rho^* \right\|_\infty \leq f\left(\rho_0(x)\right), \text{ for } x \in [0,L]$$

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➔ a zero-speed, first-order, hyperbolic PDE (with no bc's applicable)

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2nd problem: $$\frac{\partial \rho}{\partial t}(t,x) = -\sigma\left(\rho(t,x)-\rho^*\right)+\gamma\, x\left\|\rho[t]-\rho^*\right\|_\infty$$ with $\rho(t,0)=\rho^*$

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The feedback law changes the character of the PDE!
The closed-loop system is no longer a conservation law!

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The state space for the 2nd problem:

$$X_{\sigma,\gamma} := \left\{ \rho \in C^1\left([0,L];(0,\rho_{\max}]\right) : \rho(0) = \rho^*,\ f\left(\rho^*\right) + \sigma \int_0^x \left(\rho(s) - \rho^*\right) ds - \gamma \frac{x^2}{2} \left\| \rho - \rho^* \right\|_\infty \le f(\rho(x)) \right\}$$

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➔ it depends on the control parameters

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$$X_{\sigma,\gamma} := \left\{ \rho \in C^1\big([0,L];(0,\rho_{\max}]\big) : \rho(0) = \rho^*,\ f\left(\rho^*\right) + \sigma \int_0^x \left(\rho(s) - \rho^*\right) ds - \gamma \frac{x^2}{2} \left\| \rho - \rho^* \right\|_\infty \leq f(\rho(x)) \right\}$$

➔ it depends on the control parameters

➔ if $\rho_0 \in X_{\sigma,\gamma}$ then $\rho[t] \in X_{\sigma,\gamma}$ for all $t \geq 0$

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➔ we cannot handle every density profile simply by VSL (ramp metering may also be required)

Example

$$f(\rho) = \rho \exp(-\rho) \text{ with } \rho_{\max} = 1.6,\ L = 1,\ \rho^* = 0.7$$

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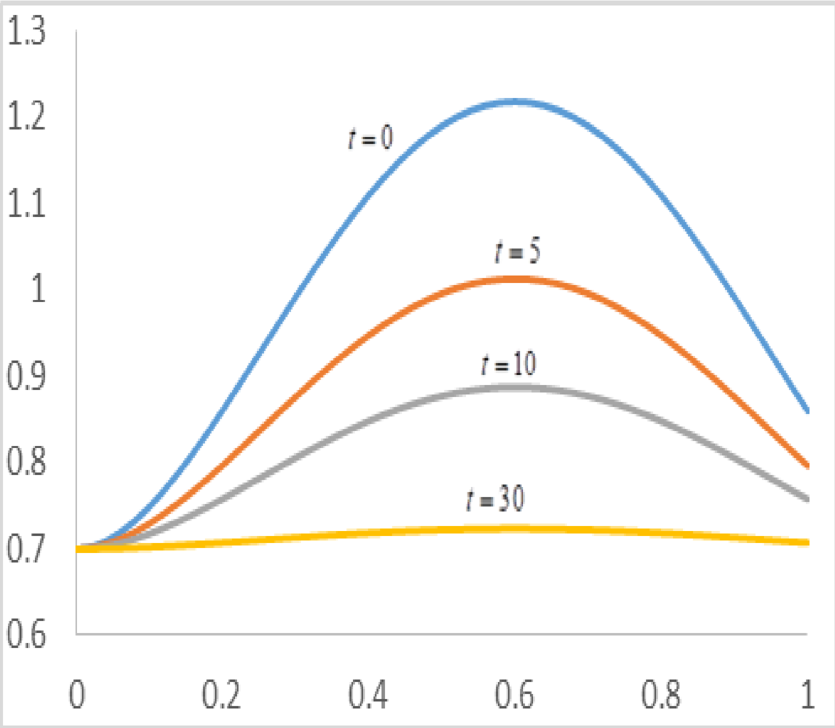
Controller parameters: $k = 0.3$, $\sigma = 0.12$, $\gamma = 0.1$

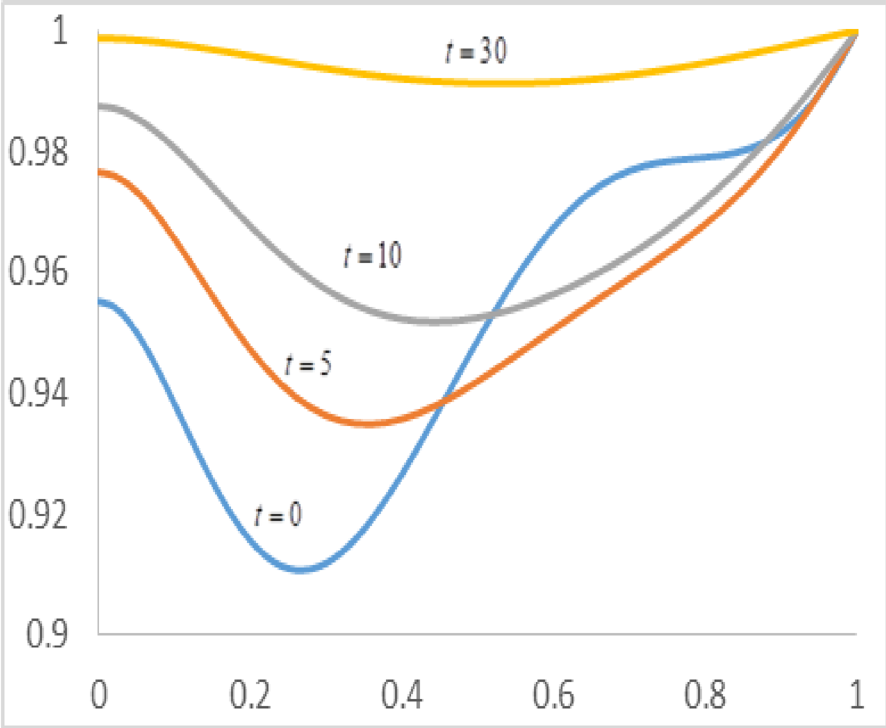
Initial condition: $\rho_0(x) = \rho^* + 4x^2(1.2 - x)^2$, $x \in [0, L]$

Example

Feedback for the 1st control problem



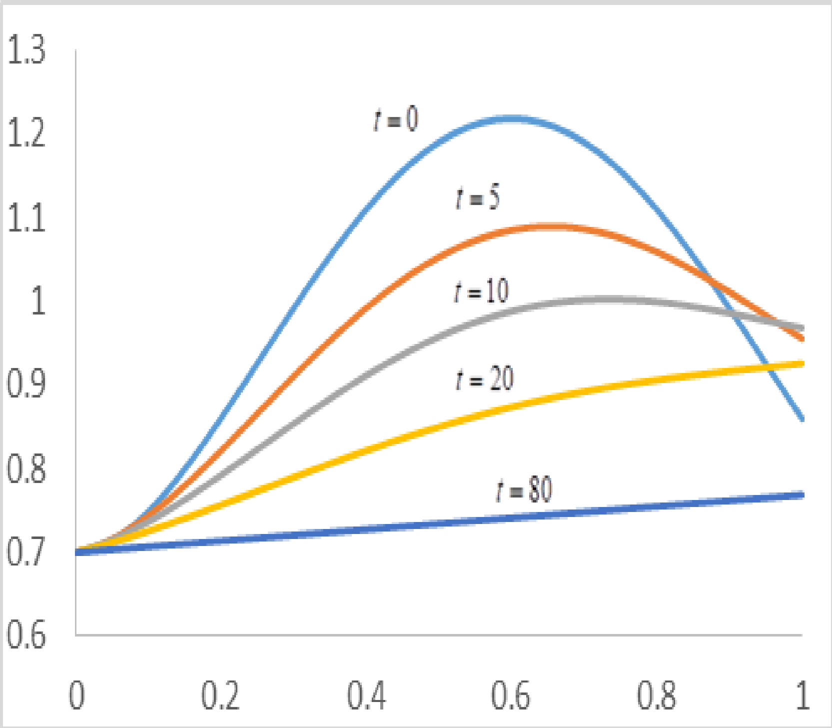
Density



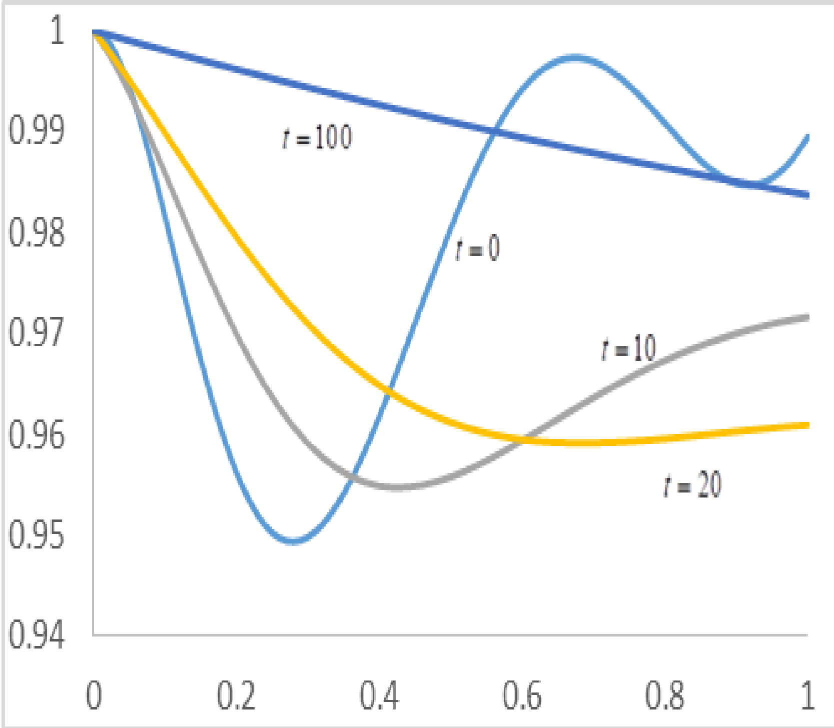
Speed limit ratio

Example

Feedback for the 2nd control problem



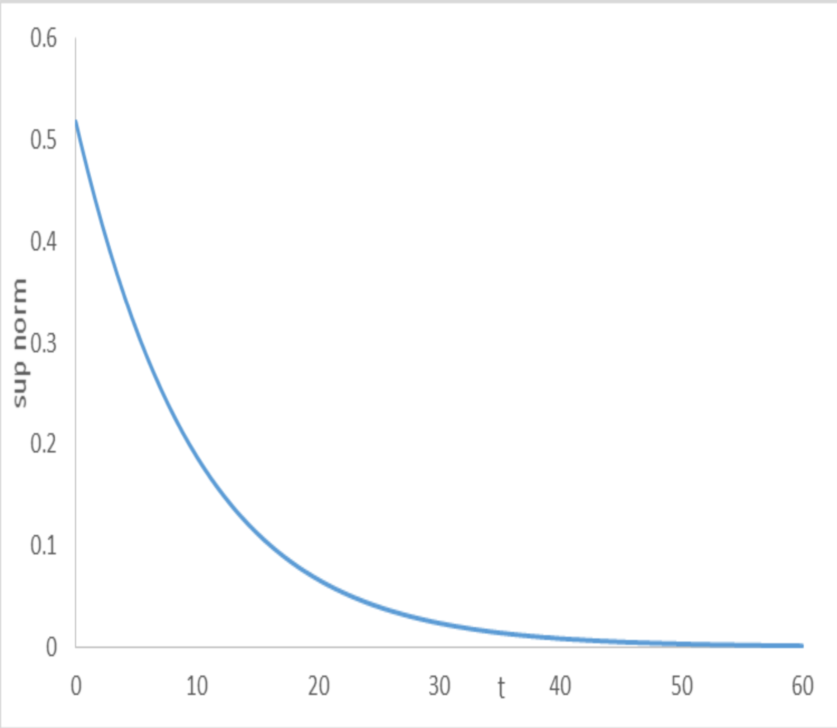
Density



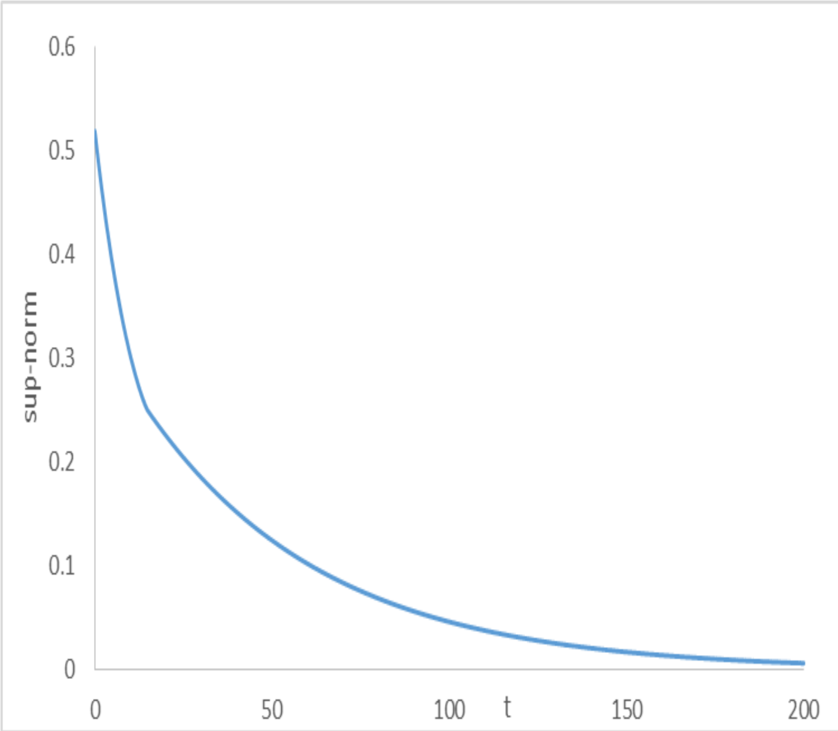
Speed limit ratio

Example

Evolution of the sup-norm of the deviation



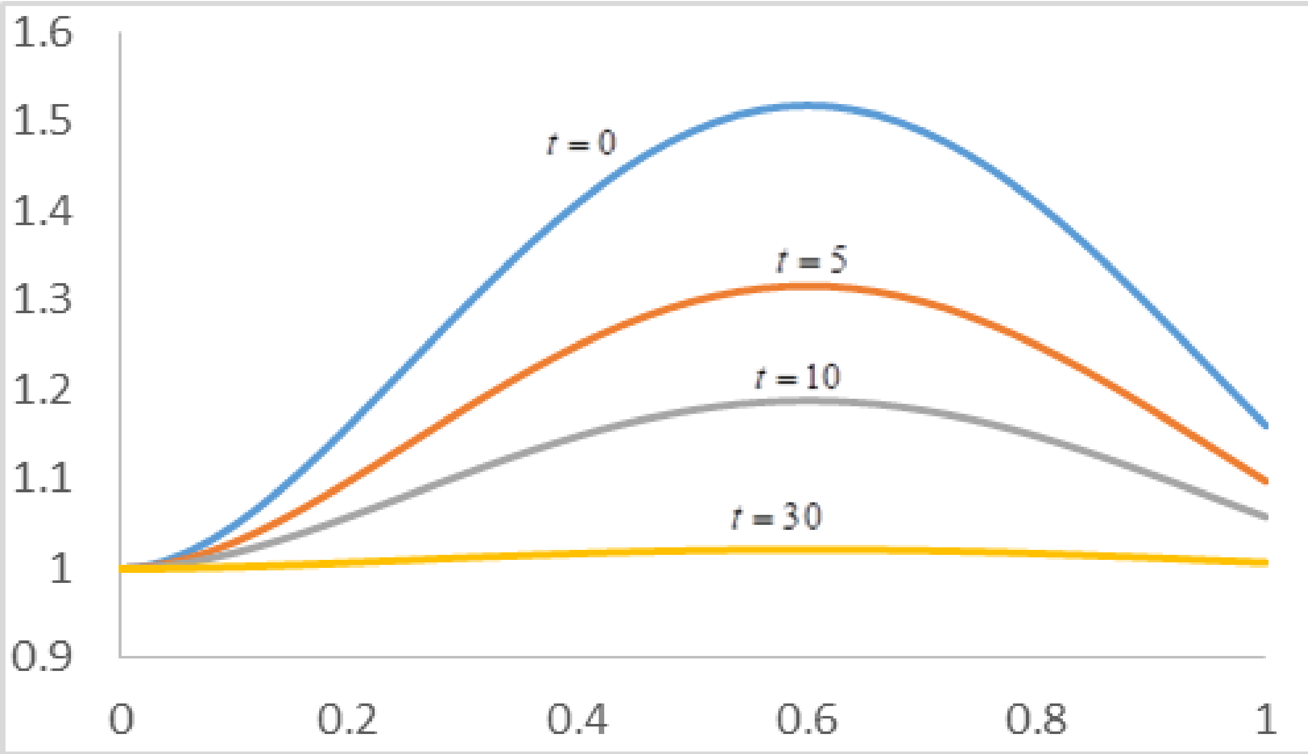
1st Control Problem



2nd Control Problem

Example

What if $\rho^* = \rho_{cr} = 1$? Only 1st control problem…



Density profiles

Example

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➔ No need to use very small speed limits: speed limit ratios greater than 90% are capable of handling congestion phenomena

➔ Intuitive solutions: speed limit ratios smaller upstream the congestion and higher downstream the congestion

Future Research

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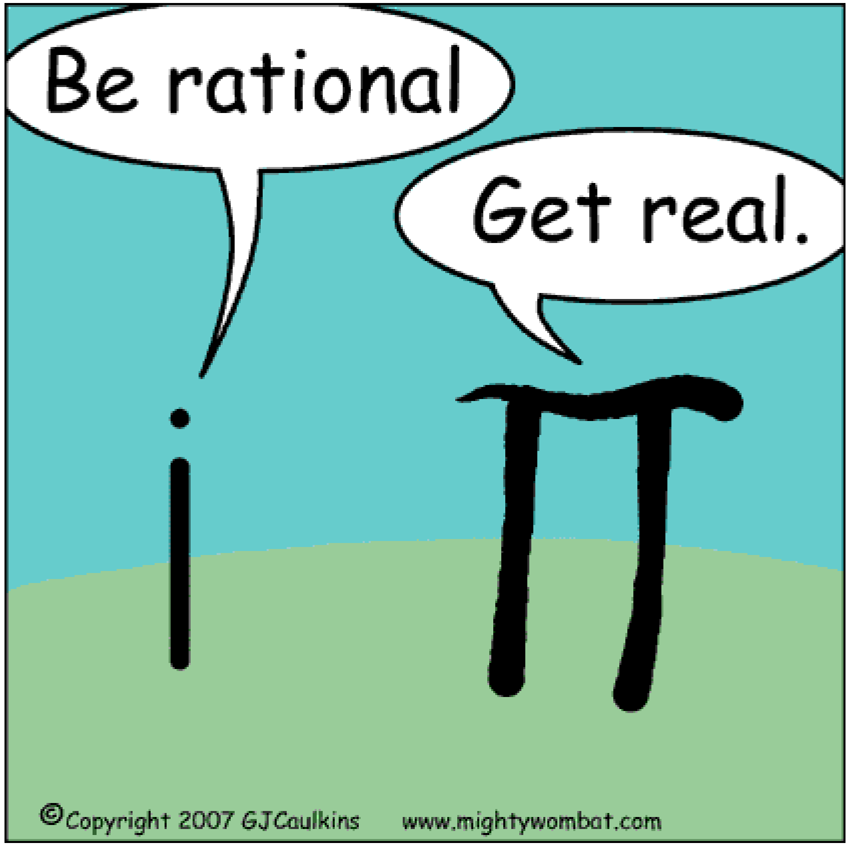
- ➔ output feedback stabilization problem
- ➔ VSL to 2nd order models?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THANK YOU!